Putnam Practice Discussions, #4

Calculus

This week a lot of our discussion centered on Calculus problems, particularly integrals. There are commonly integrals on the Putnam that severely test the use of techniques such as *substitution* and *integration by parts*.

Integration parts is suggested by the formula:

$$\int u\,dv = uv - \int v\,du.$$

The trick is to divide the given integral into two parts, a function u and a differential dv, such that the use of the parts formula is profitable.

There is a mnemonic LIATE that suggests how to do this division. The letters stand for Logarithmic, Inverse trig, Algebraic, Trig and Exponential. If the integrand may be divide into two factors corresponding to different letters in LIATE, the part that occurs to the left in LIATE should play the role of u while the part that occurs to the right should play the role of dv. This mnemonic is very effective in a very large percentage of applications of integration by parts.

Putnam 2005-A5

Evaluate $\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$.

One can apply integration by parts with $u = \ln(x+1)$ (the L part) and $dv = \frac{dx}{x^2+1}$ (the A part). Then

$$\begin{aligned}\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx &= \ln(x+1)\arctan(x)\Big|_0^1 - \int_0^1 \frac{\arctan(x)}{x+1}\,dx \\ &= \frac{\pi}{4}\ln 2 - \int_0^1 \frac{\arctan(x)}{x+1}\,dx\end{aligned}$$

Unfortunately, the integral on the other side appears just as difficult as the original integral.

This integral is particularly difficult and, as far as I know, requires either a tricky substitution or an advanced technique.

From integral calculus, we know that expressions involving x^2+1 often can be handled by the trig substitution

$$x = \tan u \qquad\qquad dx = \sec^2 u\,du$$

because $\tan^2 u + 1 = \sec^2 u$. This turns out to be a valid approach, but there are more twists ahead.

Another approach is a rational substitution. We talked about $x = \frac{1}{u}$, but that changes the integrand to ranging from 1 to ∞. There is an approach with a rational substitution that leaves the interval of integration the same.

An advanced technique that is worth knowing about is introducing a second variable:

$$f(t) = \int_0^1 \frac{\ln(tx+1)}{x^2+1}\,dx.$$

This function satisfies $f(0) = 0$, since $\ln 1 = 0$, and $f(1) =$ our desired integral. The second variable allows us to use more techniques. In particular, we can differentiate $f(t)$ with respect to t.

$$f'(t) = \int_0^1 \frac{\frac{\partial}{\partial t}\ln(tx+1)}{x^2+1}\,dx = \int_0^1 \frac{x}{(tx+1)(x^2+1)}\,dx.$$

Differentiating under the integral sign is known to be valid when the partial derivative of the integrand relative to t is continuous and the Riemann integral of the original function exists. (See [1], page 67.) This last integral may be analyzed by the method of partial fractions. Once you have that result, we can find $f(1)$ from the fundamental theorem of calculus:

$$f(1) = \int_0^1 f'(t)\,dt.$$

It's worth carrying this out to see how it plays out.

Putnam 2008-B2

Let $F_0(x) = \ln x$. For $n \geq 0$, let $F_{n+1}(x) = \int_0^x F_n(t)\,dt$. Evaluate $\lim_{n\to\infty} \frac{n!\,F_n(1)}{\ln n}$.

This is a pattern analysis problem, and with three hours available, one should simply start calculating the sequence of integrals, which in this case means integration by parts.

$$\begin{aligned} F_1(x) &= \int_0^x F_0(t)\,dt = \int_0^x \ln t\,dt \\ &= t\,\ln t\Big|_0^x - \int_0^x t\,\frac{1}{t}\,dt \\ &= x\,\ln x - \int_0^x dt = x\ln x - x \end{aligned}$$

Here we used the fact that $\lim_{t\to 0^+} t\ln t = 0$. Continuing, we have

$$\begin{aligned} F_2(x) &= \int_0^x F_1(t)\,dt = \int_0^x t\ln t - t\,dt \\ &= \int_0^x t\ln t\,dt - \frac{x^2}{2} \\ &= \frac{t^2}{2}\ln t\Big|_0^x - \int_0^x \frac{t^2}{2}\frac{1}{t}\,dt - \frac{x^2}{2} \\ &= \frac{x^2}{2}\ln x - \frac{x^2}{4} - \frac{x^2}{2} \end{aligned}$$

And again we have

$$\begin{aligned} F_3(x) &= \int_0^x F_2(t)\,dt = \int_0^x \frac{t^2}{2}\ln t - \frac{t^2}{4} - \frac{t^2}{2}\,dt \\ & \int_0^x \frac{t^2}{2}\ln t\,dt - \frac{x^3}{12} - \frac{x^3}{6} \\ &= \frac{t^3}{6}\ln t\Big|_0^x - \int_0^x \frac{t^3}{6}\frac{1}{t}\,dt - \frac{x^3}{12} - \frac{x^3}{6} \\ &= \frac{x^3}{6}\ln x - \frac{x^3}{18} - \frac{x^3}{12} - \frac{x^3}{6} \end{aligned}$$

By this point, we were prepared to guess a pattern.

$$F_n(x) = \frac{x^n}{n!}\ln x - \frac{x^n}{n!}\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + \frac{1}{1}\right).$$

The solution would have to have a proof of this pattern by the method of induction. Thus, we should state the pattern, and then prove that one integration produces the analogous formula for $F_{n+1}(x)$. We do not have to give more than one of the special cases we calculated to find the pattern. That is for our scratch paper, and not for the finished proof.

Once we have this pattern, we see that

$$F_n(1) = -\frac{1}{n!}\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + \frac{1}{1}\right).$$

Then

$$\frac{n!F_n(1)}{\ln n} = -\frac{1}{\ln n}\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + \frac{1}{1}\right).$$

It is known from the integral test approximation that the partial harmonic series shown above is asymptotic to $\ln n$. Thus, when all the pieces are in place, we will have shown the limit is -1.

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For real numbers a and b with $0 \leq a \leq b$, find

$$\int_a^b \arccos\left(\frac{x}{\sqrt{(a+b)x - ab}}\right)\, dx.$$

This horrific-looking integral may be evaluated with integration by parts and normal substitutions, but it's a hard task. Use LIATE to begin this path.

References

[1] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis.* Cambridge Univ. Press, 1927.