Putnam Practice Discussions, #3

Monthly 11449, Aug.-Sept. 2009

Revised Statement:

Find the minimum and maximum values of

$$f(a,b,c) = \frac{(a^3+b^3+c^3)^2}{(b^2+c^2)(c^2+a^2)(a^2+b^2)}$$

given that $a+b \geq c > 0$, $b+c \geq a > 0$, and $c+a \geq b > 0$.

With the new conditions, it appears the minimum value could be greater than 0.

We speculated that the maximum value of $\frac{9}{8}$ occurs when $a=b=c$.

To prove this, we might use techniques of calculus to show that any extremum must have $a=b$. By symmetry that would imply $a=b=c$.

A general principle in calculus is that a minimum or maximum occurs only at points either on the boundary of the region or at a point where the derivatives are all 0:

$$\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = \frac{\partial f}{\partial c} = 0.$$

We mentioned "logarithmic differentiation" as an approach to differentiating this complicated function. The logarithmic derivative of a nonzero function $f(x)$ is

$$LD(f(x)) = \frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$$

It gives the "percent rate of change" of a function. The product and quotient rules are much easier for logarithmic differentiation:

$$LD(f(x)g(x)) = LD(f(x)) + LD(g(x)) \qquad LD\left(\frac{f(x)}{g(x)}\right) = LD(f(x)) - LD(g(x))$$

$$LD(f(x)^n) = n\,LD(f(x))$$

Check that these agree with the product and quotient rules for ordinary derivatives. Logarithmic differentiation can also be used to solve optimization problems because $LD(f(x)) = 0$ if and only if $f'(x) = 0$.

Let's use LD_a, LD_b and LD_c to denote the partial logarithmic derivatives with respect to a, b, c, respectively. Then

$$LD_a\left(\frac{(a^3+b^3+c^3)^2}{(b^2+c^2)(c^2+a^2)(a^2+b^2)}\right) = 2\,LD_a(a^3+b^3+c^3) - LD_a(c^2+a^2) + LD_a(a^2+b^2).$$

(Why doesn't $b^2 + c^2$ appear?) Simplify these log derivatives, and set this and the formulas for LD_b and LD_c equal to 0. Then use algebra to get some information about a, b and c.

Another idea that might be useful is that the function $f(a, b, c)$ is *homogeneous* of degree 0, meaning that

$$f(ta, tb, tc) = f(a, b, c)$$

for any nonzero number t. (t would cancel out after substituting ta for a, tb for b and tc for c.) Thus, f is constant on any line through the origin.

Euler's formula and complex numbers

Euler's formula is a famous relation between exponentials of imaginary numbers and sines and cosines. It begins with the Taylor series for e^x, $\cos x$, $\sin x$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Then we use $i = \sqrt{-1}$ which is defined to satisfy $i^2 = -1$. Thus,

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\ &= \sum_{n=0}^{\infty} i^n \, \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} i^{2n} \, \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} i^{2n+1} \, \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

after separating the even and odd powers in the series. Since $i^2 = -1$, we have $i^{2n} = (i^2)^n = (-1)^n$. Thus,

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} i(-1)^n \, \frac{x^{2n+1}}{(2n+1)!} \\ &= \cos x + i \, \sin x \end{aligned}$$

This is Euler's famous formula. Some applications are

$$e^{i\pi} = \cos\pi + i\,\sin\pi = -1 \qquad e^{i(\pi/2)} = \cos(\pi/2) + i\,\sin(\pi/2) = i \qquad r\,e^{i\theta} = r\cos\theta + ir\sin\theta$$

The last one above is called *complex polar coordinates.*

Some variations of Euler's formula are

$$e^{-ix} = \cos x - i\sin x \qquad \cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

A *complex number* is a number of the form $z = x + iy$ where x, y are real numbers. We call $x = \Re(z)$ the *real part* and $y = \Im(z)$ the *imaginary part* of z. We call $\overline{z} = x - iy$ the *complex conjugate* of z, and $|z| = \sqrt{x^2 + y^2}$ is called the *absolute value* of z. We have

$$\Re(z) = \frac{1}{2}(z + \overline{z}) \qquad \Im(z) = \frac{1}{2i}(z - \overline{z}) \qquad |z|^2 = x^2 + y^2 = z\overline{z}$$

Note that $|e^{ix}| = 1$ for all real numbers x.

Last time we mentioned the formula for a *complex wave*

$$a\cos\theta + b\sin\theta = \Re\left((a - ib)e^{i\theta}\right).$$

The *complex amplitude* of this wave is $a - ib$.

Roots of Unity

Any complex number z that satisfies $z^n = 1$ is called an *n-th root of unity.* The square roots of 1 are both real ± 1. The cube roots of unity are

$$1 = e^{i0} \qquad \frac{-1+\sqrt{3}i}{2} = e^{i2\pi/3} \qquad \frac{-1-\sqrt{3}i}{2} = e^{i4\pi/3}$$

The fourth roots of unity are

$$1 = e^{i0} \qquad i = e^{i\pi/2} \qquad -1 = e^{i\pi} \qquad -i = e^{i3\pi/2}$$

In general, the n-roots of unity are

$$\omega_j = \exp\left(i\frac{2\pi j}{n}\right), \qquad 0 \le j \le n-1.$$

Since the geometric sum formula implies

$$\frac{z^n - 1}{z - 1} = z^{n-1} + z^{n-2} + \cdots + z + 1,$$

we see that the roots of the above polynomial of degree $n-1$ are ω_j for $1 \le j \le n-1$.

Some problems concerning roots of unity and Euler's formula:

603: Suppose a regular n-gon has vertices at ω_j for $0 \le j \le n-1$. What is the product of the distances from $1 = \omega_0$ to all the other vertices ω_j for $1 \le j \le n-1$?

Trig. sums: Simplify

$$\begin{aligned} &1 + \cos x + \cos 2x + \cdots + \cos nx \\ &\sin x + \sin 2x + \cdots + \sin nx \end{aligned}$$

634: Let **a**, **b** be the vectors

$$\begin{aligned} \mathbf{a} &= \left[\cos\left(\frac{2\pi}{n}\right), \cos\left(\frac{4\pi}{n}\right), \cos\left(\frac{6\pi}{n}\right), \ldots, \cos\left(\frac{2n\pi}{n}\right)\right] \\ \mathbf{b} &= \left[\sin\left(\frac{2\pi}{n}\right), \sin\left(\frac{4\pi}{n}\right), \sin\left(\frac{6\pi}{n}\right), \ldots, \sin\left(\frac{2n\pi}{n}\right)\right] \end{aligned}$$

Show that the intersection of the plane spanned by **a** and **b** in $\mathbb{R}^n$ with the unit cube centered at the origin in $\mathbb{R}^n$ is a regular $2n$-gon.