

Putnam Practice Discussions, #2

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Find the minimum and maximum values of

$$f(a, b, c) = \frac{(a^3 + b^3 + c^3)^2}{(b^2 + c^2)(c^2 + a^2)(a^2 + b^2)}$$

given that $a^2 + b^2 \geq c^2$, $b^2 + c^2 \geq a^2$, and $c^2 + a^2 \geq b^2$.

In our discussion, we saw that $a = -b$, $c = 0$ gives a minimum value of 0.

We speculated that the maximum value of $\frac{9}{8}$ occurs when $a = b = c$. To prove this, we might use techniques of calculus to show that any extremum must have $a = b$. By symmetry that would imply $a = b = c$.

A general principle in calculus is that a minimum or maximum occurs only at points either on the boundary of the region or at a point where the derivatives are all 0:

$$\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = \frac{\partial f}{\partial c} = 0.$$

This is a slightly messy function to differentiate. You can alleviate that mess partly by considering changing the function in a way that the derivative is easier to calculate, and nonetheless the new equations give the same solutions as the above equation. (one word: logarithm.) Try calculating these derivatives, and then try to use these equations to prove $a = b = c$ at any maximum point.

Putnam 2008-B3

This problem asked for the maximum radius of a circle contained entirely in the hypercube $[-\frac{1}{2}, \frac{1}{2}]^4$.

The key here is to come up with a description of circles in 4-dimensional space. We came up with

$$\mathbf{x}(\theta) = \mathbf{a} \cos \theta + \mathbf{b} \sin \theta$$

for $0 \leq \theta < 2\pi$, where $\mathbf{a} = [a_1, a_2, a_3, a_4]$ and $\mathbf{b} = [b_1, b_2, b_3, b_4]$ are vectors with the same magnitude $|\mathbf{a}| = |\mathbf{b}| = r$ and which are perpendicular, which means their dot product $\mathbf{a} \cdot \mathbf{b} = 0$.

In order for this circle to be contained in the hypercube, we need each component formula to satisfy:

$$|a_j \cos \theta + b_j \sin \theta| \leq \frac{1}{2}.$$

We stated a formula for the maximum of this combination of trig. functions, namely, $\sqrt{a_j^2 + b_j^2}$.

These were the ingredients for showing that the maximum radius is $r = \frac{1}{\sqrt{2}}$.