

Putnam Practice Discussions

Greetings to everyone who likes math problems! These are notes on our meetings preparing for the Putnam Math Competition on December 5, 2009. We began meeting on Wednesday September 23 in MS 428, and will continue to do so for the rest of the fall term from 1:30 to 2:20 PM.

Putnam 2008-A1

Let $f(x, y)$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all x, y, z . Prove there is a function $g(x)$ such that $f(x, y) = g(x) - g(y)$.

Comments: This is a classic problem where very little is known about the function. We cannot use any methods of calculus since nothing is known about whether or not they can be applied to $f(x, y)$. All we have available is the equation $f(x, y) + f(y, z) + f(z, x) = 0$. We are allowed to substitute different values in for x, y, z and thereby learn things about the function $f(x, y)$. Everyone should do this problem and write out its answer in detail as excellent practice.

Putnam 2008-A2

This was a game between Alan and Barbara where they take turns putting a number in an entry of a 2008×2008 matrix. The first player Alan wins at the end if the determinant of the matrix is nonzero, and the second player Barbara wins if the determinant is 0. The question is which player is guaranteed to win under optimal play.

Comments: The key here is to use some information about determinants. The main thing to know is that if a square matrix has two identical rows it must have determinant equal to 0.

Putnam 2008-B1

The question here is what is the maximum number of rational points (x, y) (so both x and y are fractions) which can lie on a circle with an irrational center.

You can turn this around by asking if we have enough rational points can we prove a the center of the circle is also rational.

A key fact about circles is that there is a unique circle through any three non-collinear points in the plane. Three points determine a circle.

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This was a problem in the elite Problems column of the *American Mathematical Monthly*.

Find the minimum and maximum values of

$$\frac{(a^3 + b^3 + c^3)^2}{(b^2 + c^2)(c^2 + a^2)(a^2 + b^2)}$$

given that $a^2 + b^2 \geq c^2$, $b^2 + c^2 \geq a^2$, and $c^2 + a^2 \geq b^2$.

In our discussion, we saw that $a = -b$, $c = 0$ gives a minimum value of 0.

We speculated that the maximum value of $\frac{9}{8}$ occurs when $a = b = c$. To prove this, we might use techniques of calculus to show that any extremum must have $a = b$. By symmetry that would imply $a = b = c$.

Putnam 2008-B3

This problem asked for the maximum radius of a circle contained entirely in the hypercube $[-\frac{1}{2}, \frac{1}{2}]^4$.

The key here is to come up with a description of circles in 4-dimensional space.