

1 The Paper

A required part of Math 6490 is a project on the topics of this course, with a written report of about 10 pages, depending on the appendices, and giving a brief oral presentation on this report at the end of term. The project can be anything on the subjects of this term including roughly: rigid motions and similarities, tilings and patterns, Möbius transformations and complex geometry, the topology of surfaces, the enumeration of groups, and lots more. The project could be historical, mathematical or a programming project.

Here are some guidelines.

- The finished report should be neatly presented, with your written expositions typed where possible. Take care of niceties such as sectioning, page numbers, references and citations, charts or diagrams, etc. I will shortly pass out a checklist on formatting issues to consider.
- I will allow you to work in pairs, if you and a classmate wish to work together on a project.
- The total workload over the course of the term should be something around 25 hours, assuming you are efficient about those hours. If you're working in a pair, the workload should be something more than that, probably 15 or 20 hours per person.
- By March 11, I will ask for a brief paragraph stating your plan for your work.

2 Brief Suggestions of Topics

Classification of Archimedean tilings: This would be a math project explaining how to determine all the different semi-regular tilings, meaning a tiling by regular polygons where every vertex has the same arrangement of regular tiles around it. An alternative would be a programming project which would prepare either PostScript or Maple programs for all the tilings.

Classification of discrete groups of rigid motions of the plane: This would be a mathematical determination of all these groups: including the rosette groups, strip groups and wallpaper groups. A general reference for all the tiling projects is the big book of tilings [Grünbaum and Shephard, 1987].

Classification of rigid motions in three dimensions: We have classified the rigid motions in two dimensions as translations, rotations, reflections and glide reflections. This project would be a study of the different types of rigid motions in three dimensions.

Discrete groups of rotations in three dimensions: This would be a study of rotations of the two-dimensional sphere, leading to a listing of all the possible discrete groups of rotations. It could also involve drawing the tilings of the sphere corresponding to these discrete groups, and the associated regular polyhedra. A short and beautiful paper on the subject is [Senechal, 1990]. You could also look at Klein's *Lectures on the Icosahedron* [Klein, 1956].

Discrete groups of rigid motions in three dimensions: This would be a somewhat advanced project, as the variety of discrete groups in three dimensions is much greater than that in two dimensions. This classification is part of the science of *crystallography*, which is of huge practical importance in chemistry, solid state physics and materials science. One project of the right scale would be just a brief overview of the crystallographic groups.

Aperiodic tilings: This would be further reading and study of the aperiodic tilings such as the Penrose tiling. there are many possible directions that one can follow here, but for the purpose of this course it would be particularly relevant to study those tilings that come out of an inflation-and-subdivision recipe. The original Penrose article is [Penrose, 1979]. Martin Gardner's columns on it are in the book [Gardner, 1997]. He refers to this theory in his book about the possibilities of artificial intelligence, *The Emperor's New Mind* [Penrose, 1989]. There is a nice book [Senechal, 1995] on the modern theory of quasicrystals that came out of Penrose's discovery. The big book [Grünbaum and Shephard, 1987] of tilings has an extensive chapter on aperiodic tilings.

***L*-systems:** This would be a study of a general pattern of recursively defined geometric structures first formulated by Aristid Lindenmayer. Our creation of Penrose tilings was an example of such a system. The project would consist of learning what an *L*-system is, and possibly how to create them in PostScript or some other language that allows recursive programming. A good book on this is *The Algorithmic*

Beauty of Plants [Prusinkiewicz and Lindenmayer, 1990]. Another is [Prusinkiewicz and Hanan, 1989]. There is a conference proceedings [Rozenberg and Salomaa, 1992] on lots of different aspects of Lindenmayer systems.

Formal rewriting systems: A **rewriting system** consists of some symbols and some rules for rewriting combinations of the symbols. For example the Fibonacci system consists of the symbols $\{a, b\}$ and the rules $a \rightarrow b$ and $b \rightarrow ba$. If we apply these rules over and over, we get *generations*: $a, b, ba, bab, babba, babbabab$, etc. This project would be a description of some of these systems and the properties they possess. See some of the references in the previous project and also [Choffrut, 1992].

Circle Packings: A packing of circles is an arrangement of disjoint circles in the plane. We can associate a combinatorial graph to such a packing by making each circle a vertex and by connecting two vertices by an edge if the corresponding circles are tangent. This graph is called the **nerve** of the packing. This project would be a study of basic properties of circle packings including when a circle packing can be found whose nerve is a prescribed graph. A survey is given in the article [Stephenson, 2003]. A method of calculating and drawing circle packings is described in [Collins and Stephenson, 2003].

Conway's Tiling Groups: This describes a group-theoretic method invented by Conway for deciding if certain regions of the plane may be tiled by a standard shape, such as a domino or a triomino. The main reference is the article [Thurston, 1990]. The goal would be to learn the method and to carry out in some examples.

Automatic groups: This is described in Chapter 11 of [Mumford et al., 2002]. An **automatic group** is like a special kind of rewriting system that applies particularly to groups. This project would be to learn the basic definition of a finite state automaton and to calculate some automata for some simple groups.

Schattschneider's problem on classifying patterns: Doris Schattschneider described a very interesting problem connected with groups of rigid

motions in her talk at OSU last fall. It's concerned with some mathematics carried out by the Dutch artist M. C. Escher on the possible patterns he could construct with certain prescribed symmetries. The project would be to learn about the statement of the problem, and to describe some simple results and maybe work on some parts which are still undone. The main reference is Schattschneider's book [Schattschneider, 1990].

Schwarzian differential operator and Möbius transformations: We will learn about the Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ which are the angle-preserving transformations of the complex plane (more precisely the Riemann sphere). There is an amazing ordinary differential operator \mathcal{S} , called the **Schwarzian**, which has the property that if $\mathcal{S}(f) = 0$, then $f(z)$ is a Möbius transformation. Moreover, if $\mathcal{S}(f) = \mathcal{S}(g)$, then $f(z) = \frac{ag(z)+b}{cg(z)+d}$ for some a, b, c, d . This project would be about learning what the Schwarzian is and why it works in this way. A basic reference for this and the next two projects is the book by Lester Ford [Ford, 1951].

The hyperbolic metric on the upper half-plane: The Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ where a, b, c, d are all real have the property that they map points in the upper half-plane $\mathbb{H} = \{\Im z > 0\}$ back into the upper half-plane. There is a notion of distance between two points in the upper half-plane such that these Möbius transformations are precisely the distance-preserving transformations of the upper half-plane. This project would involve learning the definition of this distance, called the **hyperbolic metric**, and some of its basic properties. An important point to learn is what are the paths of shortest distance between two points (they are not straight lines in general).

Triangle groups: A triangle group is defined by beginning with a triangle and considering all possible combinations of reflections in the three sides of the triangle. The condition for this to be a discrete group of rigid motions depends on the angles of the triangle only. The **hyperbolic** triangles are constructed in the same way except that the sides of the triangle are allowed to be circular arcs, and in place of reflections we consider the inversions in the three circular arcs. If we apply the resulting group of transformations to the original triangle, we get a tiling by triangles. Two examples are Figures 12.3 and 12.4

of [Mumford et al., 2002]. This project would consist of a study of triangle groups possibly culminating in some computer drawings of the associated tilings.

The construction of surfaces: Conway's ZIP proof: We have discussed how two-dimensional surfaces can be constructed by taking a polygon with an even number of sides and pasting together pairs of the edges. Can all surfaces be constructed in this way? This project would be a study of a proof of that theorem by an ingenious argument of John Conway consisting of dividing a surface first into triangular pieces with zippers along the edges. The main reference is the beautifully illustrated article [Francis and Weeks, 1999].

3 Details on Certain Projects

Classification of Archimedean tilings

The first step is classifying all the **vertex figures** that can occur. These are arrangements of regular polygons around a vertex with interior angles adding up to 360° . The interior angle of a regular n -gon is $180 - 360/n$ degrees. By working out the resulting equation and considering the positive integers ≥ 3 that can occur, we can show there are exactly 21 possible vertex figures.

The next step is to see how vertex figures would piece together. Usually just positioning a couple will show whether or not it is possible to find a tiling with that vertex figure. PostScript would be very helpful for experimentation on this. In this way, we will wind up with exactly all the Archimedean tilings.

Classification of discrete groups of rigid motions

This is a mathematical rediscovery project. We know the rigid motions of the plane can be classified as

- Translations
- Rotations
- Reflections
- Glide Reflections

As described in Chapter 1, a collection of these is a **group** if it is closed under composition and under taking inverses.

A group of transformations is **discrete** if there is a circular disk D such that the image $T(D)$ of this disk under any transformation in the group, other than the identity, is disjoint from D .

The goal of this project is to describe all the possible discrete groups of rigid motions in a systematic way. An advanced reference is Chapter 5, Sections 1 through 4, of Michael Artin's *Algebra* [Artin, 1995]. But we will indicate the major steps in the classification, and the project will consist of filling in the details.

Here are some major steps. Let G be a discrete group. What we want is a list of possibilities for the constructions of G .

- First we have to distinguish the orientation-preserving transformations (translations and rotations) from the orientation-reversing transformations (reflections and glide reflections). It's important to observe that the composition of two transformations of the same type is always orientation-preserving, while that of two transformations of different types is orientation-reversing. Based on that lemma, prove:

The subcollection G_0 of all orientation-preserving transformations in G is also a subgroup of G (namely closed under inverses and composition).

Then prove:

Either $G = G_0$ or there is an orientation reversing transformation T in G such that G consists of G_0 and TG_0 (meaning the composition of T with every element of G_0).

- Now turn to the classification of the orientation-preserving groups G_0 . These consist entirely of rotations or translations. We can do the same sort of reduction again by considering the subcollection G_1 of translations in G_0 . Since the composition of any two translations is also a translation, G_1 is again a subgroup of G_0 .

Using complex numbers, each translation T is of the form $T(z) = z + a$ for some complex constant a . Here $z = x + iy$ is a variable point in the plane. Thus, the collection G_1 of translations corresponds in a one-to-one way to the complex numbers a . Moreover, if $T_1(z) = z + a_1$ and $T_2(z) = z + a_2$, then $T_1 \circ T_2(z) = z + (a_1 + a_2)$, and $T_1^{-1}(z) = z - a_1$. Thus, G_1 corresponds to a collection of complex numbers a which are closed under addition and subtraction. The condition of discreteness means there is a circular disk which does not contain any two points of the form z and $z + a$ for some $a \neq 0$ in the collection. That means there is a distance $\epsilon > 0$ such that $|a| \geq \epsilon$ for all $a \neq 0$ in the collection.

Prove that any collection of complex numbers is one of

1. $\{0\}$ (just the zero point)
2. $\{ma \mid \text{for any integer } m\}$, for a fixed nonzero complex number a . (In other words all multiples of a .)

3. $\{ma + nb \mid \text{for any integers } m, n\}$, for complex numbers a, b which are not multiples of one another. (In other words, all integral linear combinations of a and b .)

The group is call **rank 0, 1, or 2**, respectively.

- Next we have to consider the rotations that can be added to a group of translations as described in the previous step. For the group to be discrete, prove the rotations must all be by a multiple of $2\pi/n$ radians.

If there are no nonzero translations, the n may be arbitrary, but all the rotations must be around the same point, which can be taken to be 0. This is called the **cyclic group**.

If there are nonzero translations, then prove that n has to be 1, 2, 3, 4, or 6. If the rank is 1, prove that $n = 1$ or 2.

If the rank is 2, prove that $n = 3, 4$ or 6 occurs only for special choices of the **basic** translations a and b . That will complete the list of orientation-preserving discrete groups.

- Lastly, we have to decide when we can add a single reflection or glide reflection to one of the groups G_0 that we have listed. Here it is important to think about conjugacy. If T is the orientation-reversing transformation that we add, and S is any translation or rotation, then $T \circ S \circ T^{-1}$ is also translation or rotation in the group G . So the reflection or glide reflection that we add must conjugate our existing translations and rotations among themselves. That severely limits the choice of reflection or glide reflection that we can add. Once we analyze those choices, we'll have the complete list of wallpaper groups.

Rigid transformations in 3 or higher dimensions

First construct some of the easy rigid transformations in three dimensions. For instance, if \mathbf{x} is a variable vector and \mathbf{a} is a constant vector, we have the **translation** $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$.

Then, if T is any rigid motion, show that there is a translation S and a rigid motion R with the property that $R(0) = 0$ (i.e. it fixes the origin) such that $T = S \circ R$. This is just the same as in two dimensions, and reduces us to the the problem of classifying rigid motions that fix the origin.

Use the same argument from Chapter 1 (Project 1.2) to prove that any such rigid motion must be a linear map. That leads us to expressions of the form $R(\mathbf{x}) = A\mathbf{x}$, where A is a 3×3 matrix. (By the way, all this works in any dimension, not just two and three).

Assuming that R is distance-preserving, derive a condition that the matrix A must satisfy. This is based on the linear algebra that $|\mathbf{x}|^2 = \mathbf{x}^T \mathbf{x}$, where $|\mathbf{x}|$ is the usual Euclidean length of the vector \mathbf{x} , thought of as a column, and \mathbf{x}^T is the transpose of \mathbf{x} as a row vector. Such matrices are called **orthogonal**.

Prove that any orthogonal matrix is either a rotation by a certain angle around a certain line through the origin (called the *axis of rotation*), or a reflection in a plane through the origin. This is based on eigenvalue theory.

After that, you will have that any rigid motion is a composition of a translation and either a rotation about a line through the origin, or a reflection through a plane through the origin. The tricky part after this is to classify which of these are conjugate to one another. Think about the fixed points, lines, planes. Make three-dimensional pictures with Maple of how the rigid transformations operate.

Schattschneider's problem on classifying patterns

Schattschneider's problem can first be stated for a tiling of squares. This is symmetric under the group generated by translations $T_1(x, y) = (x + 1, y)$, $T_2(x, y) = (x, y + 1)$, the rotation by 90° given by $R(x, y) = (y, -x)$ and the reflection $W(x, y) = (x, -y)$.

Suppose we have a collection of four figures A , B , C and D which are pictures to place on a square. How many ways can we assign one of these four to all the squares in the tiling in such a way that the assignment is invariant under the translations generated by $T_1^2(x, y) = (x + 2, y)$ and $T_2^2(x, y) = (x, y + 2)$? We say two assignments are the same if we can apply one of the symmetries of the square tiling to one assignment and obtain the other.

This is trickier than it sounds, because it depends on the relations between the four figures, i.e. whether or not some of them are rotations or reflections of the other.

Schwarzian differential operator and Möbius transformations

The Schwarzian derivative is given by the formula

$$\mathcal{S}(f) = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2$$

It's miraculous properties are that for any Möbius transformation $m(z)$ we have

$$\mathcal{S}(m \circ f) = \mathcal{S}(f)$$

and

$$\mathcal{S}(f \circ m)(z) = \mathcal{S}(f)(m(z)) m'(z)^2$$

It takes some very careful application of the chain rule just to verify these formulas. It is more of a miracle that if two functions f, g have the same Schwarzian, then there must be a Möbius transformation m such that $f = g \circ m$. This comes from the theory of linear differential equations. A basic reference is [Ford, 1951], which has many other wonderful things in it as well.

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