

**FINAL EXAM**  
**MATH 3013 SECTION 002, SPRING 2009**

INSTRUCTOR: WEIPING LI

**Print Name and Student #**

**SHOW WORK FOR CREDIT !!! SHOW WORK FOR CREDIT !!!**

(1) (10pts) (a) Verify the matrix  $A = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$  is orthogonal.

(b) Let  $u = q_1 - 2q_2 + 2q_3$  for an orthonormal basis  $\{q_1, q_2, q_3\}$  in  $R^3$ . Find  $\|u\|$ .

(2) (10pts) Let  $A$  be a  $4 \times 4$  matrix with  $\det A = 5$ . Find the following.

(i)  $\det(2A^T + A^T)$

(ii)  $\det(A^T \cdot A)$

(iii)  $\det(A^{-3})$ .

(3) (10pts) Find a basis for the orthogonal complement  $W^\perp$  in  $\mathbb{R}^4$  of the subspace  $W = \text{sp}\{(1, 2, 2, 1), (0, 1, 2, -1)\}$ .

(4) (15pts) Find an orthogonal basis from the basis  $\{(1, 0, 1), (0, 1, 2), (2, 1, 0)\}$  for  $\mathbb{R}^3$  by using the Gram-Schmidt process.

(5) (10pts) Given an orthonormal basis for  $W = \text{sp}\{v_1, v_2, v_3\}$  in  $R^4$ , where  $v_1 = (1, 1, 1, 1)$ ,  $v_2 = (-1, 1, -1, 1)$  and  $v_3 = (1, -1, -1, 1)$ . Find the projection of  $b = (1, 2, -1, -2)$  on  $W$ .

(6) (10pts) Let  $u$  and  $v$  be vectors in an inner-product space with  $\|u\| = 3$ ,  $\|v\| = 5$ . If  $u$  and  $v$  are perpendicular, then find  $\langle u + 2v, 3u + v \rangle$ .

(7) (10pts) Determine all values  $b_1, b_2, b_3$  such that  $b = (b_1, b_2, b_3)$  lies in the span of  $\mathbf{v}_1 = (1, 0, 0)$ ,  $\mathbf{v}_2 = (1, 2, 1)$  and  $\mathbf{v}_3 = (-1, -2, -1)$ .

(8) (15pts) (a) Determine whether the matrix  $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$  is invertible, and find its inverse.

(b) Express  $A$  as a product of elementary matrices.

(9) (10pts) Find the coordinate vector of the polynomial  $x^3 - 4x^2 + 3x + 7$  relative to the ordered basis  $B_1 = \{(x - 2)^3, (x - 2)^2, (x - 2), 1\}$ .

- (10) (15pts) Find (i) the rank of  $A$  and the nullity of  $A$ , (ii) a basis for the row space, (iii) a basis for the column space of  $A = \begin{pmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 4 & -2 & 0 \\ 3 & 11 & -4 & -1 & 6 \\ 2 & 5 & 3 & -4 & 0 \end{pmatrix}$ .

- (11) (15pts) Let  $T$  be a linear transformation such that  $T(-1, 1) = (2, 1, 4)$  and  $T(1, 1) = (-6, 3, 2)$ .
- (i) Find  $T(x, y)$ .

- (ii) Using (i), compute  $T(-1, 1)$ .

(12) (15pts) Let  $T : P_3 \rightarrow P_3$  be defined by  $T(p(x)) = (x+2)p'(x)$ , where  $p'(x)$  is the derivative of  $p(x)$ . Let  $B = \{x^3, x^2, x, 1\} = B'$  be an ordered basis for  $P_3$ .

(i) Find the matrix representation  $A$  of  $T$  relative to  $B, B'$ .

(ii) Use  $A$  to compute  $T(x^3 - 2x^2 + 3x - 4)$ .

(13) (10pts) Find the  $x_2$  **ONLY** by **using Cramer's rule** for

$$5x_1 - 2x_2 + x_3 = 1$$

$$x_2 + x_3 = 0$$

$$x_1 + 6x_2 - x_3 = 4.$$

(14) (10pts) Find the characteristic polynomial of  $\begin{pmatrix} -1 & 0 & 1 \\ -7 & 2 & 5 \\ 3 & 0 & 1 \end{pmatrix}$ .

(15) (10pts) Let  $A = \begin{pmatrix} -1 & -1 & -2 \\ 8 & -11 & -8 \\ -10 & 11 & 7 \end{pmatrix}$ . The characteristic polynomial of  $A$  is given by  $(\lambda - 1)(\lambda + 3)^2$ .

(i) Find the algebraic multiplicities of  $\lambda_1 = 1$  and  $\lambda_2 = -3$ .

(ii) Find the geometric multiplicities of  $\lambda_1 = 1$  and  $\lambda_2 = -3$ .

(i) Is the matrix  $A$  diagonalizable? Must provide a reason for credit!

- (16) (10pts) A matrix  $A = \begin{pmatrix} -3 & 10 & -6 \\ 0 & 7 & -6 \\ 0 & 0 & 1 \end{pmatrix}$  has eigenvalues  $\lambda_1 = -3$  with eigenvector  $v_1 = (1, 0, 0)$ ,  $\lambda_2 = 1$  with  $v_2 = (1, 1, 1)$  and  $\lambda_3 = 7$  with  $v_3 = (1, 1, 0)$ . Compute  $A^{2009}$ .

- (17) (15pts) Solve the system of linear differential equations:

$$x_1' = x_1 - 3x_2 + 3x_3$$

$$x_2' = -5x_2 + 6x_3$$

$$x_3' = -3x_2 + 4x_3,$$

where the matrix  $\begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$  has eigenvalues  $1, 1, -2$ .