

# CONCORDANCE OF SEIFERT SURFACES

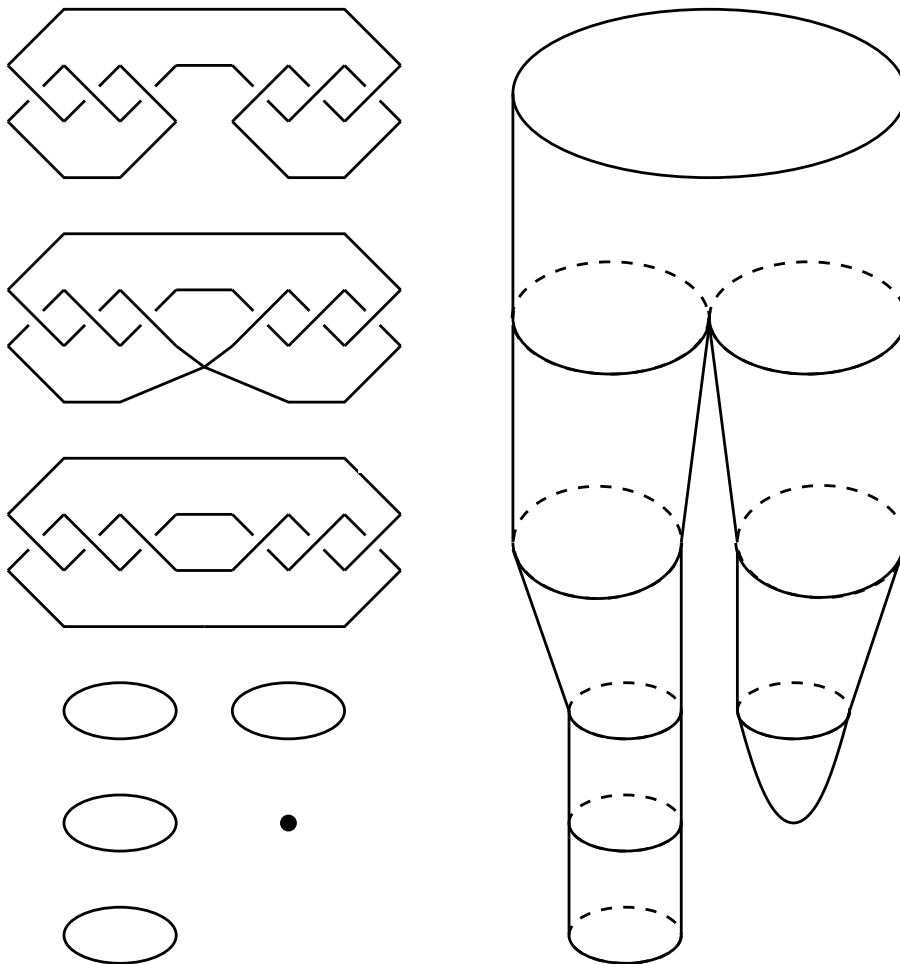
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A bibliography is on the last slides.

## CONCORDANCE OF KNOTS

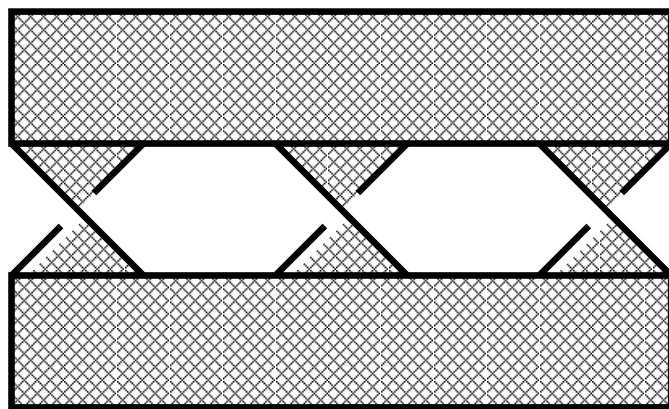
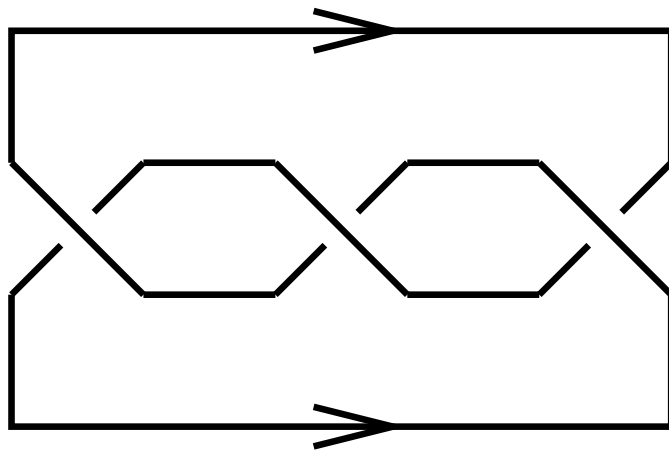
Oriented knots  $K, K'$  in  $S^3$  are **(smoothly) concordant** if there is a smoothly embedded annulus  $A$  in  $S^3 \times [0, 1]$  joining  $S^3 \times \{0\}$  to  $S^3 \times \{1\}$  with  $\partial A = K' - K$ .



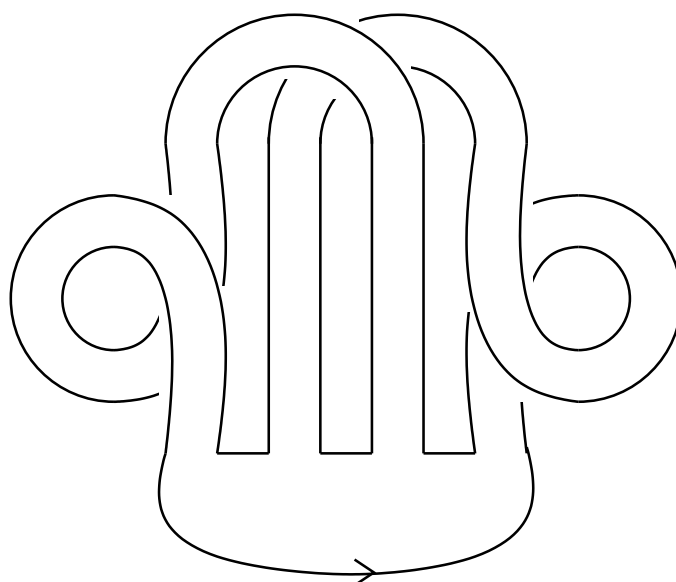
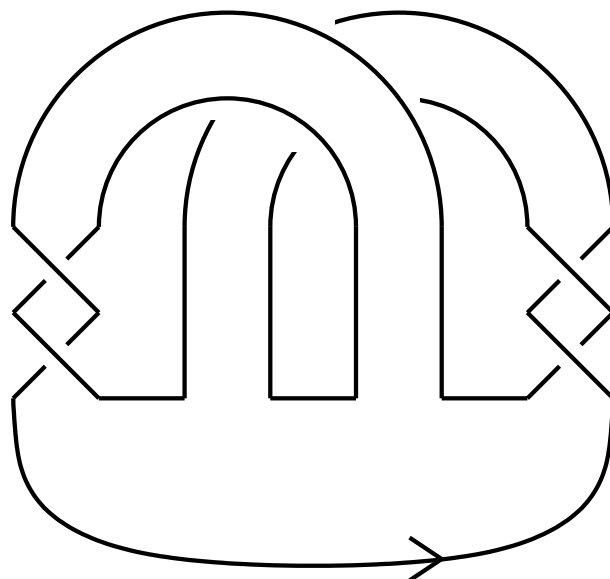
## SEIFERT SURFACES

A **Seifert surface** for  $K$  is a compact, oriented surface  $F$  with  $\partial F = K$ .

**Seifert's algorithm:** Take a projection of  $K$ . Replace crossings with shortcuts to get **Seifert circles**. They bound disjoint disks. Join them by twisted bands at the crossings.

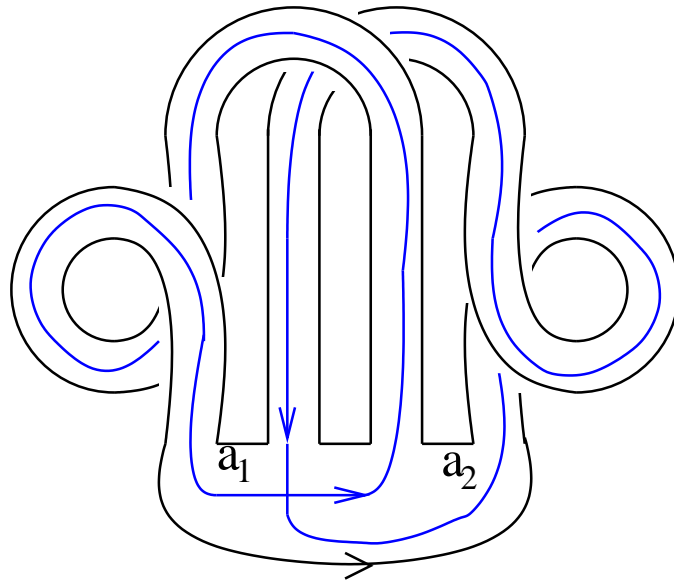


Regard  $F$  as a disk with bands.



## SEIFERT MATRICES

Choose a basis  $\{a_1, \dots, a_{2g}\}$  of oriented simple closed curves for  $H_1(F)$ .



Let  $a_i^+$  be  $a_i$  pushed off  $F$  in the positive normal direction. The **Seifert matrix**  $V$  has  $v_{i,j}$  equal to the linking number of  $a_i$  and  $a_j^+$ .

Exercise: In this example

$$V = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

## SOME KNOT INVARIANTS

The **exterior** of  $K$  is  $X = S^3 - \text{int } N(K)$ .

$$H_1(X) = \mathbb{Z} = \langle t \rangle. \quad \Lambda = \mathbb{Z}[t, t^{-1}].$$

$p : \widetilde{X} \rightarrow X$  is the infinite cyclic covering.

The **Alexander module** is  $A = H_1(\widetilde{X}, \Lambda)$ .

Then  $(V^T - tV)$  is a presentation matrix for  $A$ .

The **Alexander polynomial** is given by  $\Delta(t) = \det(V^T - tV)$ .

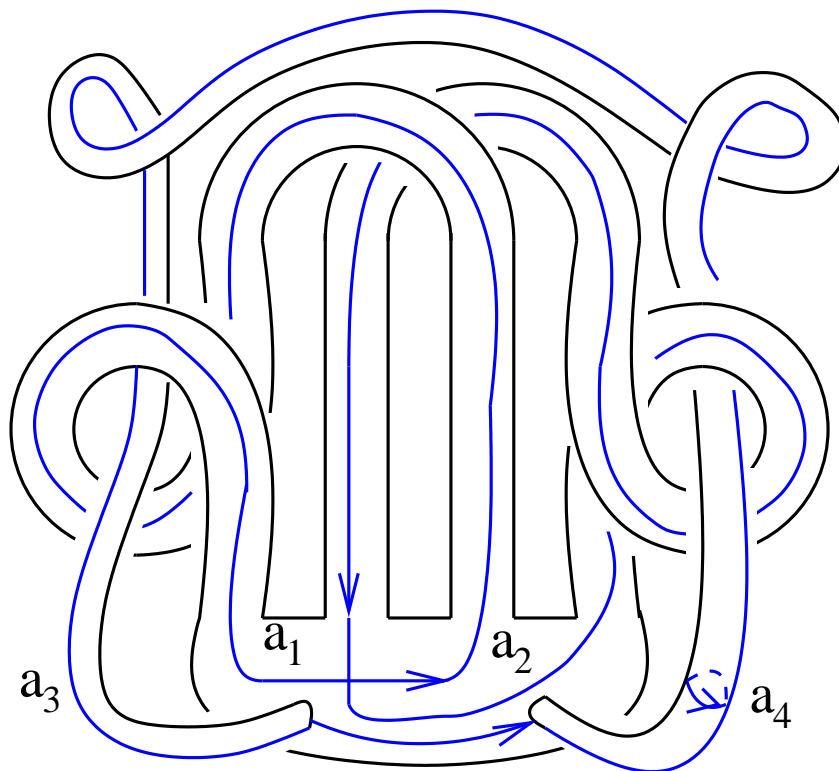
Note:  $A$  is determined by  $G/G^{(2)}$ , where  $G = \pi_1(X)$  and  $G^{(2)}$  is the second commutator subgroup.

The **signature** of  $K$  is  $\sigma(V^T + V)$ .

Change of basis changes  $V$  by an integral congruence  $P^tVP$ ,  $P \in GL(2, \mathbb{Z})$ .

What does changing the Seifert surface do?

Consider tubing:



$$V' = \left[ \begin{array}{cc|cc} -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ \hline -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

In general, tubing gives

$$\left[ \begin{array}{c|cc} V & x & 0 \\ \hline y & m & 1 \\ 0 & 0 & 0 \end{array} \right] \text{ or } \left[ \begin{array}{c|cc} V & x & 0 \\ \hline y & m & 0 \\ 0 & 1 & 0 \end{array} \right]$$

These are congruent, respectively, to the **elementary expansions**

$$\left[ \begin{array}{c|cc} V & x & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right] \text{ and } \left[ \begin{array}{c|cc} V & 0 & 0 \\ \hline y & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$$

Seifert matrices  $V$  and  $V'$  are **S-equivalent** if one can pass from  $V$  to  $V'$  by a sequence of congruences, elementary expansions, and **elementary reductions**.

Any two Seifert surfaces  $F$  and  $F'$  for a knot  $K$  are **stably equivalent**: There is a Seifert surface  $F''$  which is isotopic to surfaces obtained by successive tubing of  $F$  and successive tubing of  $F'$ .

S-equivalence preserves invariants derived from  $V$  such as signature, Alexander module, etc.

Two knots  $K$  and  $K'$  are **S-equivalent** if there exist respective Seifert matrices  $V$  and  $V'$  which are S-equivalent.

## HYPERBOLICITY

Suppose  $M$  is a compact, connected, orientable 3-manifold such that  $\partial M$  consists of tori.  $M$  is **hyperbolic** if the universal covering space of  $\text{int } M$  is hyperbolic 3-space with the covering translations isometries.

A **hyperbolic knot** is a knot whose exterior is excellent.

**[Friedl 07]:** Every knot is S-equivalent to a hyperbolic knot of arbitrarily large volume.

**[M. 83]:** Every knot is concordant to a hyperbolic knot.

## OUTLINE OF FRIEDL'S PROOF

1. Blanchfield pairing  $\beta : A \times A \rightarrow Q(\Lambda)/\Lambda$ .  
 $Q(\Lambda)$  is the field of fractions of  $\Lambda$ .

**[Trotter 73]** or **[Kearton 75]**+**[Levine 70]**:  
Knots are S-equivalent if and only if they have isometric Blanchfield pairings.

2. **[Kawauchi 89]**: Every knot  $K$  has a hyperbolic almost identical imitation  $K'$  of arbitrarily large volume.

This implies that there is a map  $f : (S^3, K') \rightarrow (S^3, K)$  inducing an isomorphism on the knot groups modulo their second commutator subgroups.

3. **[Friedl 07]**: Such a map induces an isometry of Blanchfield pairings.

Alternatively, **[Kawauchi 94]** shows that such a map implies S-equivalence.

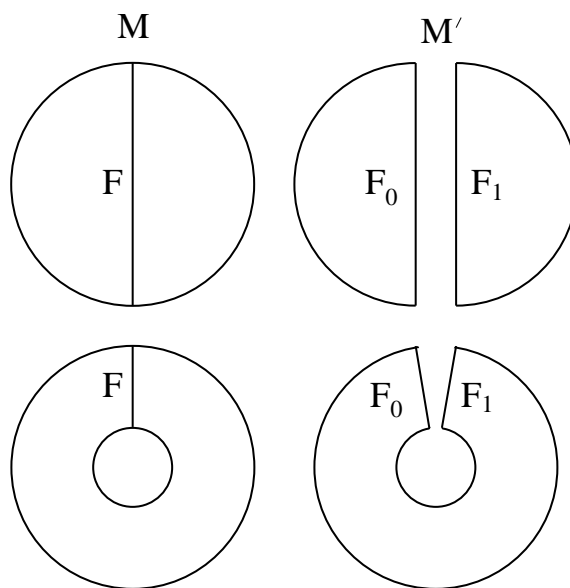
## EXCELLENCE

Let  $M$  be a compact, connected, orientable 3-manifold.  $M$  is **excellent** if it is irreducible,  $\partial$ -irreducible, anannular, and atoroidal, contains an incompressible surface, and is not a 3-ball.

An **excellent 1-manifold** is a properly embedded 1-manifold in a 3-manifold whose exterior is excellent.

**Thurston's Hyperbolization Theorem:** If  $\partial M$  consists of tori and  $M$  is excellent, then  $M$  is hyperbolic.

**Gluing Lemma:** Suppose  $F$  is a compact, properly embedded surface in  $M$ . Let  $M'$  be  $M$  split along  $F$ . Let  $F_0$  and  $F_1$  be the copies of  $F$  in  $M'$ . If each component of  $M'$  is excellent,  $F_0$ ,  $F_1$ , and  $\partial M' - (F_0 \cup F_1)$  are incompressible in  $M'$ , and each component of  $F$  has  $\chi < 0$ , then  $M$  is excellent.



Is a geometric proof of Friedl's result hidden in the literature? Suspects:

**[Kanenobu 86]:**  $K'$  hyperbolic, ribbon concordant to  $K$ , and has isomorphic Alexander module.

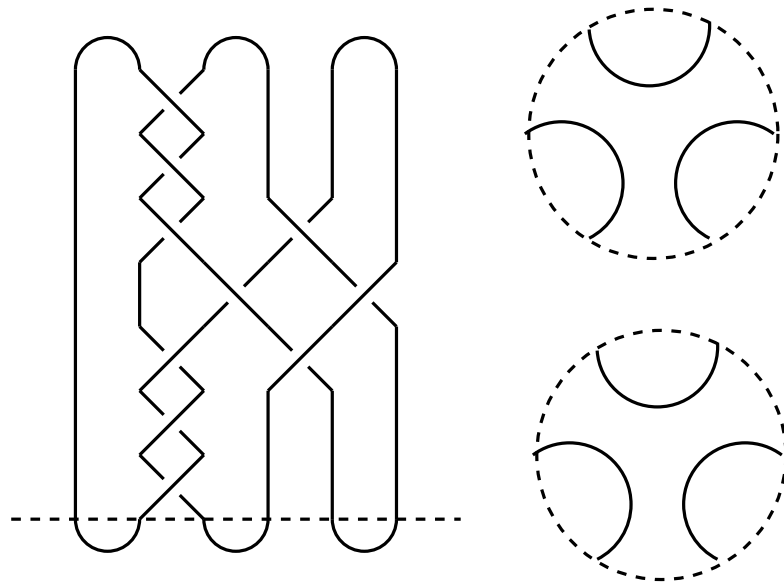
**[Kawauchi 89]:** Hyperbolic almost identical imitations.

**[Silver-Whitten 05]:** Hyperbolic covering knots:  $f : (S^3, K') \rightarrow (S^3, K)$  inducing an epimorphism of knot groups,  $K'$  hyperbolic, ribbon concordant to  $K$ , and has isomorphic Alexander module.

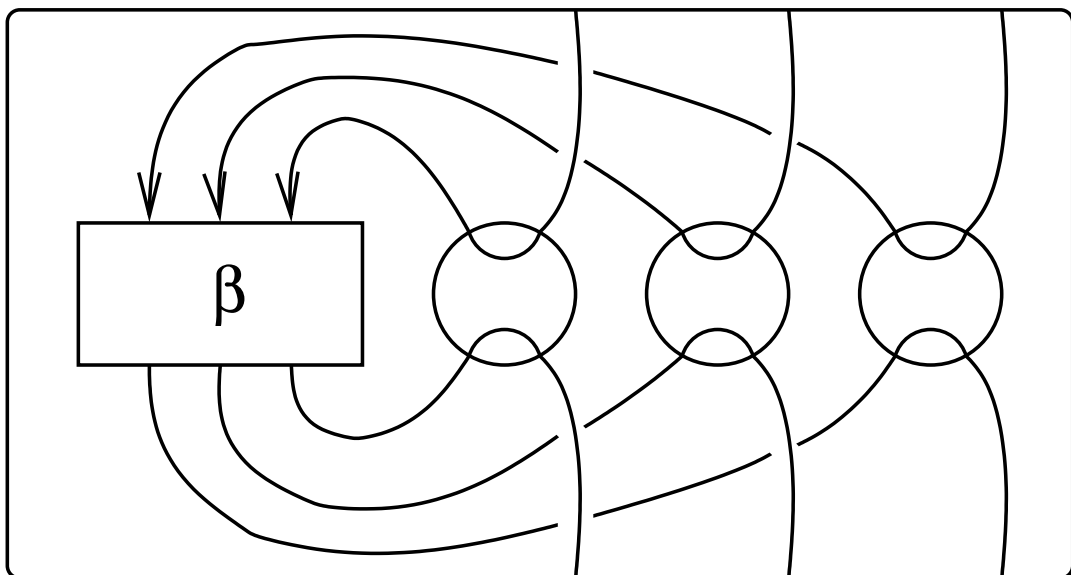
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Common strategy: Choose a special projection, choose special trivial 2-tangles, replace them by special non-trivial 2-tangles.

Kawauchi uses the **Kanenobu construction**:  
 Start with an  $n$ -bridge presentation of  $K$ .

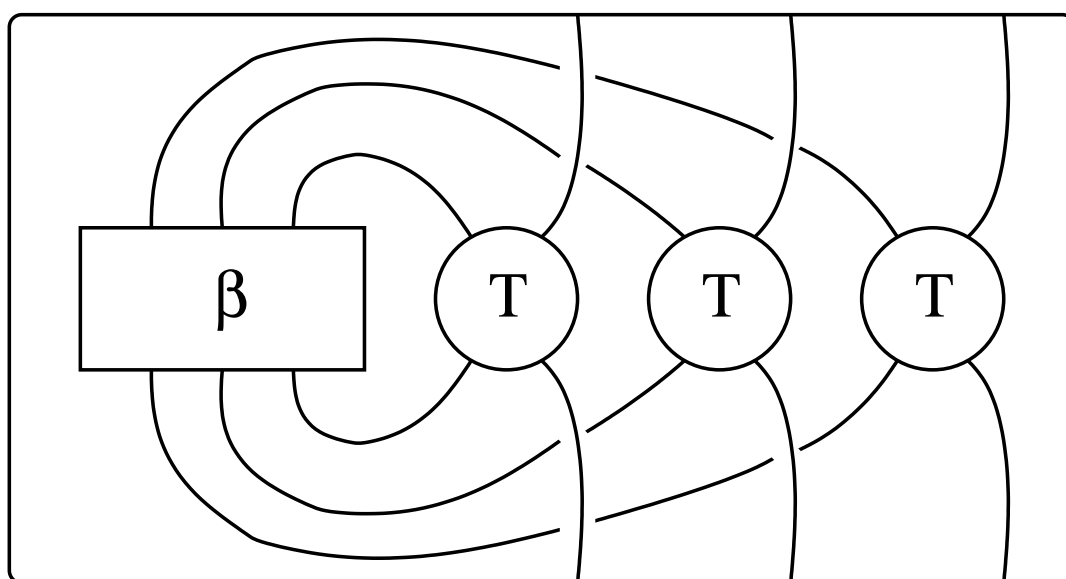


Take this diagram of the trivial  $n$ -tangle, where  $\beta$  is a certain special pure  $n$ -braid.



A prime 2-tangle is **simple** if its exterior contains no incompressible tori.

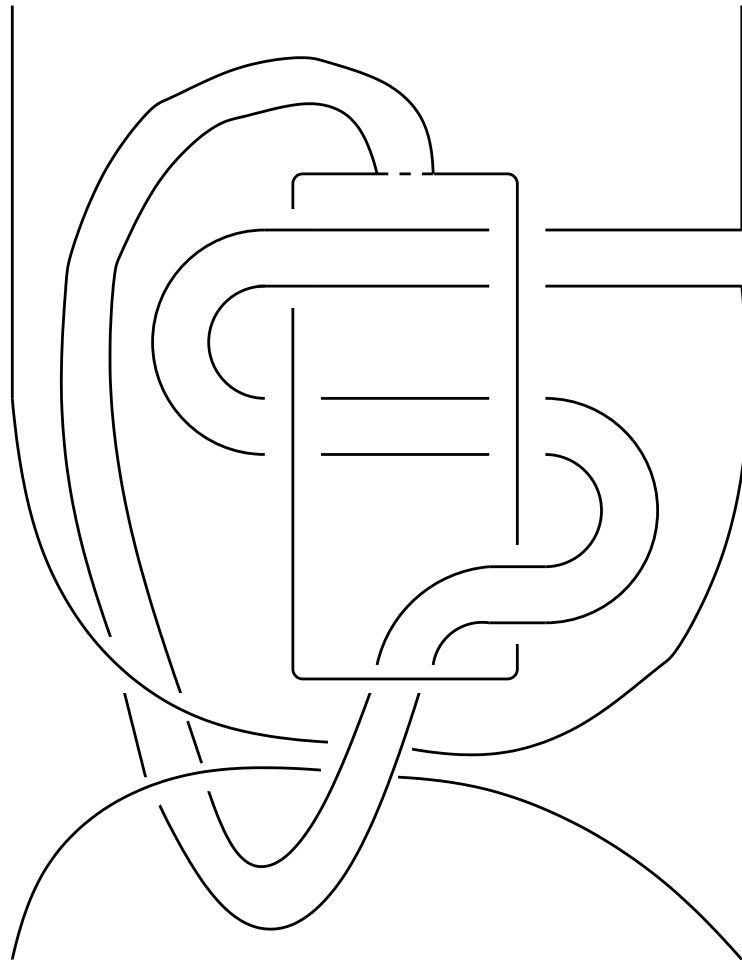
Replace the trivial 2-tangles by copies of a simple 2-tangle  $T$ .



Kanenobu shows that the new  $n$ -tangle is excellent.

By the Gluing Lemma the new knot is excellent, hence hyperbolic.

Kawauchi uses the following simple tangle.

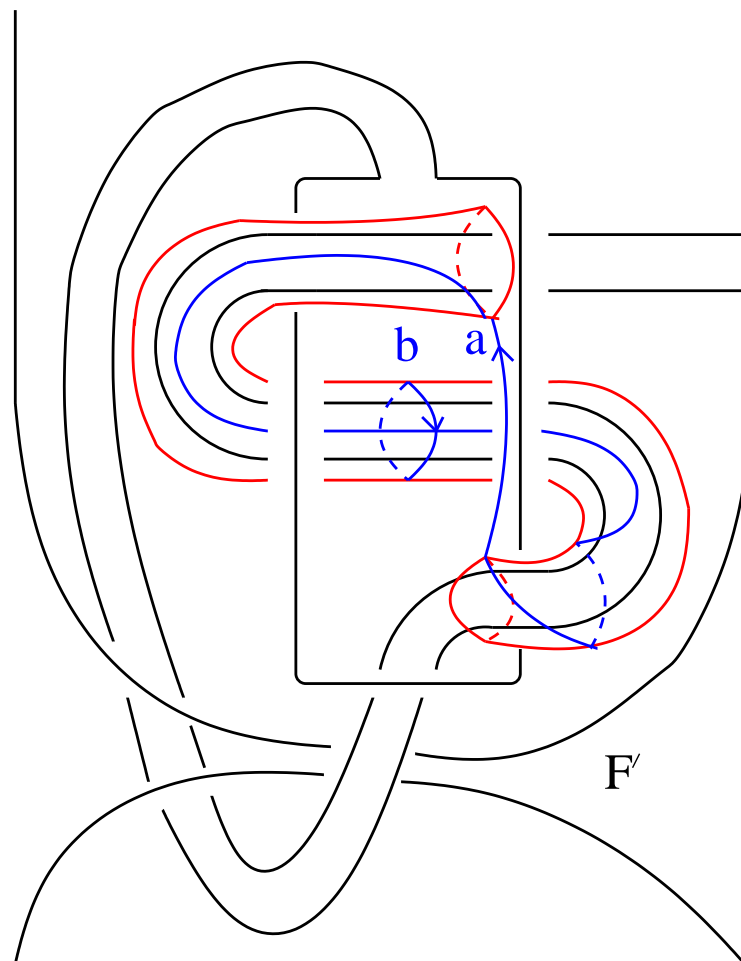


Cut along the dashed line to see a concordance.

There is an easy proof of S-equivalence:

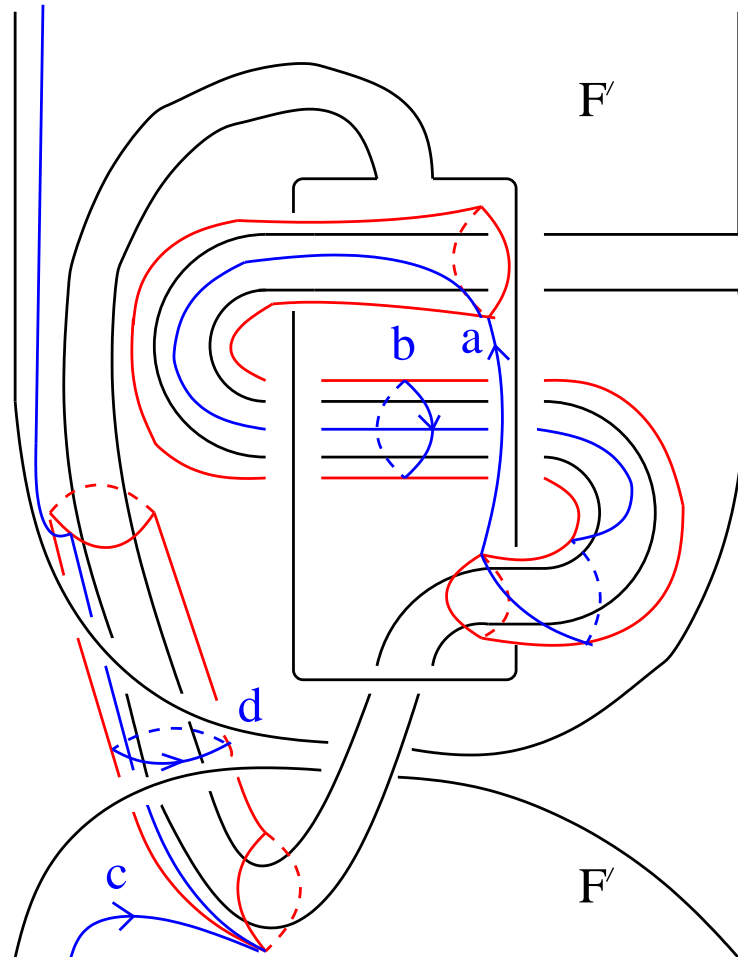
The arcs of the trivial 2-tangles lie on the same Seifert circle.

Case 1: The Seifert disk meets each tangle ball in a disk.



$$V' = \left[ \begin{array}{c|cc} V & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

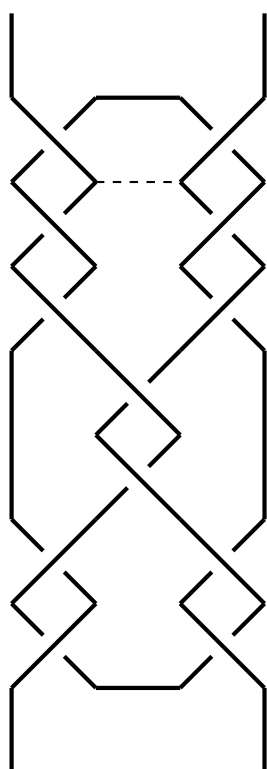
Case 2: The Seifert disk meets each tangle ball in two disks.



$$V' = \left[ \begin{array}{c|ccc|cc} V & 0 & 0 & x & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline y & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \sim \left[ \begin{array}{c|ccc|cc} V & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

**Remark:** A similar argument shows S-equivalence for the Silver-Whitten construction.

Kanenobu uses the K-T grabber.



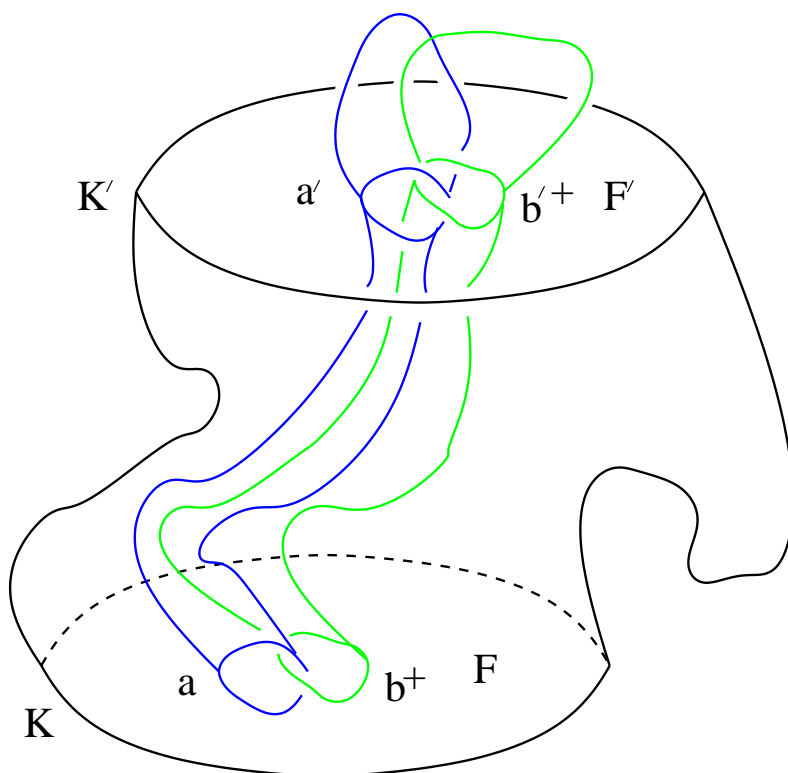
Cut along the dashed line to see a concordance.

**Question:** Does this give an S-equivalence?

## NEW PROOF

Seifert surfaces  $F, F'$  for knots  $K, K'$  are **concordant** if there is a smooth embedding of  $F \times [0, 1]$  in  $S^3 \times [0, 1]$  with  $F \times \{0\} = F$  and  $F \times \{1\} = F'$ .

$F$  and  $F'$  have a common Seifert matrix, and so  $K$  and  $K'$  are S-equivalent.



**Theorem:** If  $F$  is not a disk, then  $F$  is concordant to  $F'$  such that

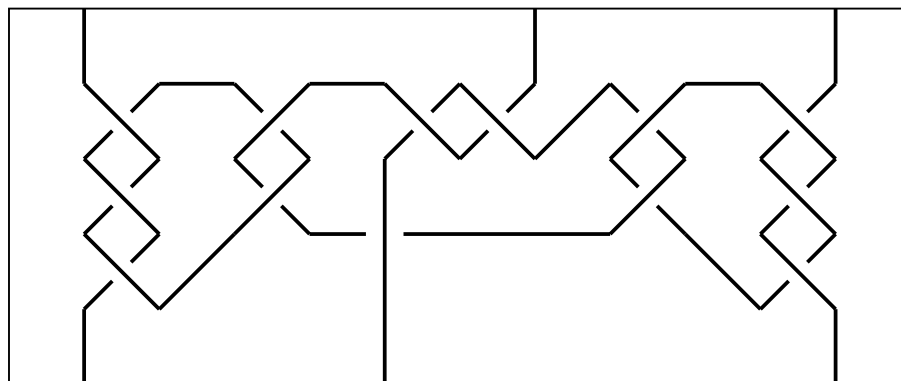
1.  $K'$  is hyperbolic,
2.  $S^3 - K'$  has arbitrarily large volume, and
3. There is a map of pairs

$$f : (S^3, K') \rightarrow (S^3, K)$$

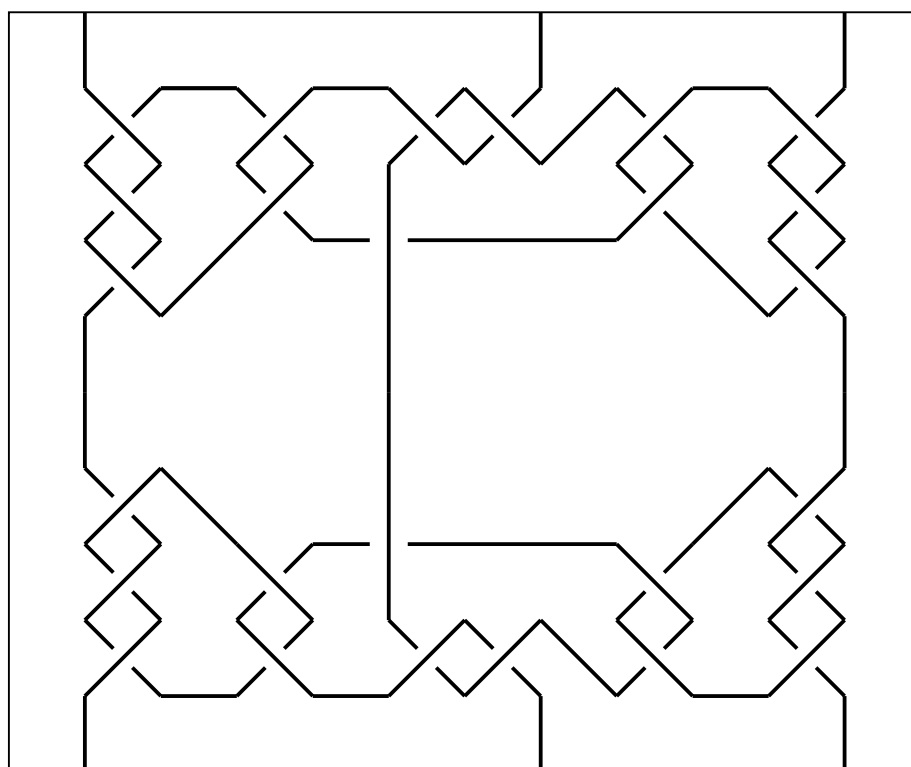
which induces an epimorphism

$$f_* : \pi_1(S^3 - K') \rightarrow \pi_1(S^3 - K)$$

**Proof of Theorem:** Take any excellent  $n$ -tangle, e.g. [M. 83].

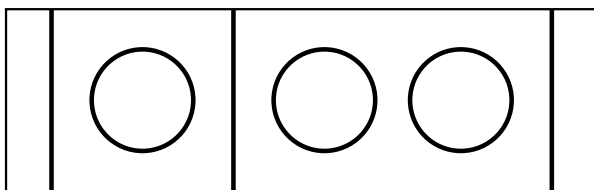
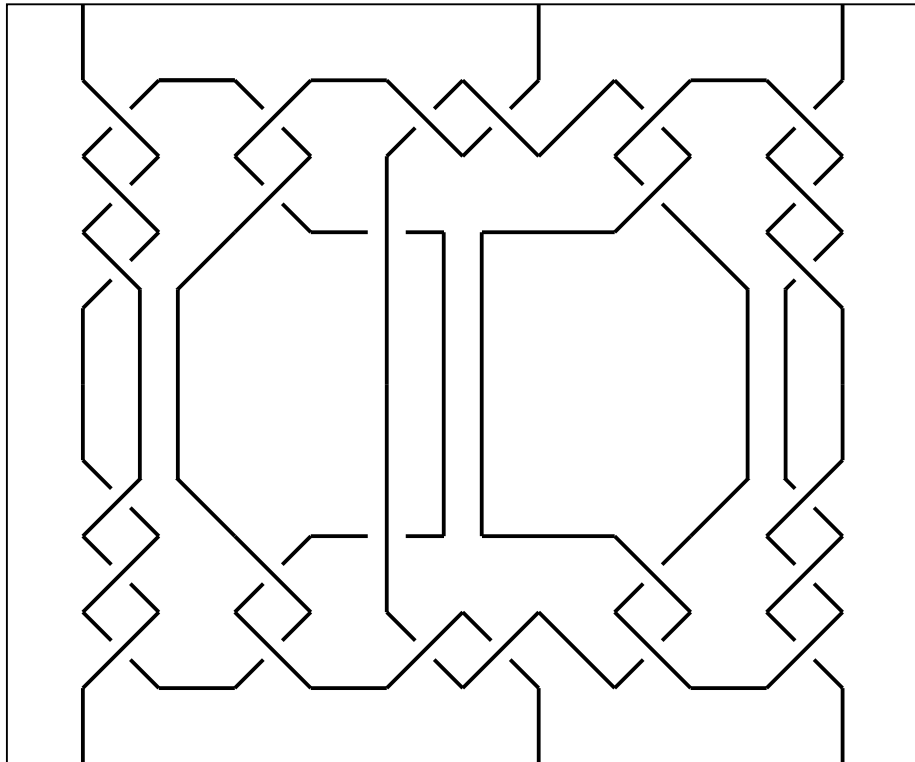


Take its sum  $T_n$  with its reflection.

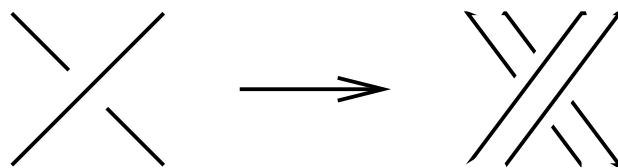


$T_n$  is also excellent.

This is concordant to the trivial  $n$ -tangle.

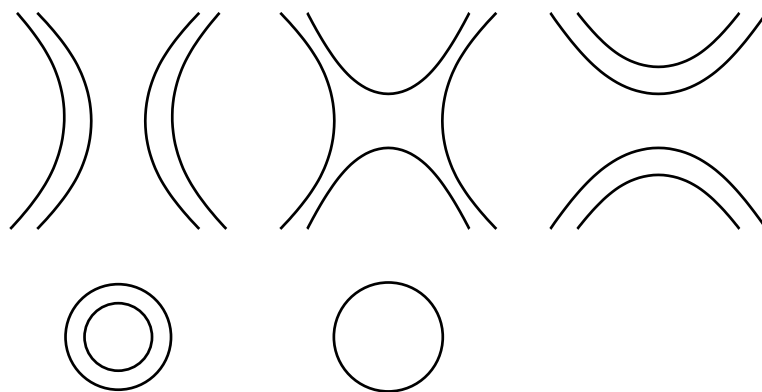


Replace each arc of  $T_n$  by a band to get an  $n$ -band tangle  $B_n$ .

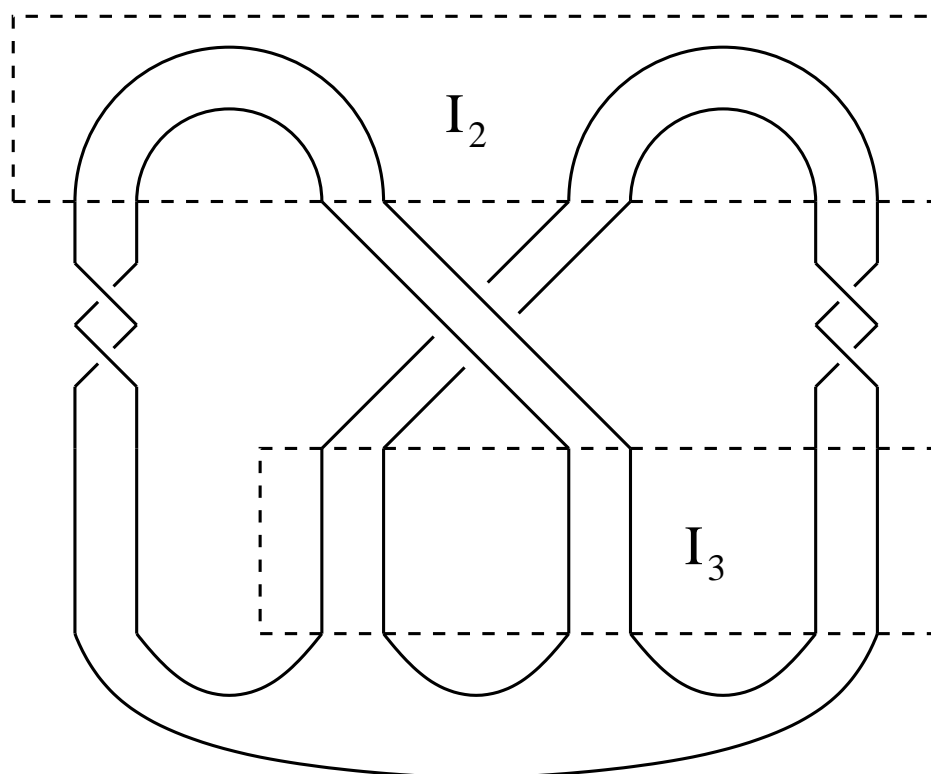


The exterior of  $B_n$  is homeomorphic to the exterior of  $T_n$ .

$B_n$  is concordant to the trivial  $n$ -band tangle.

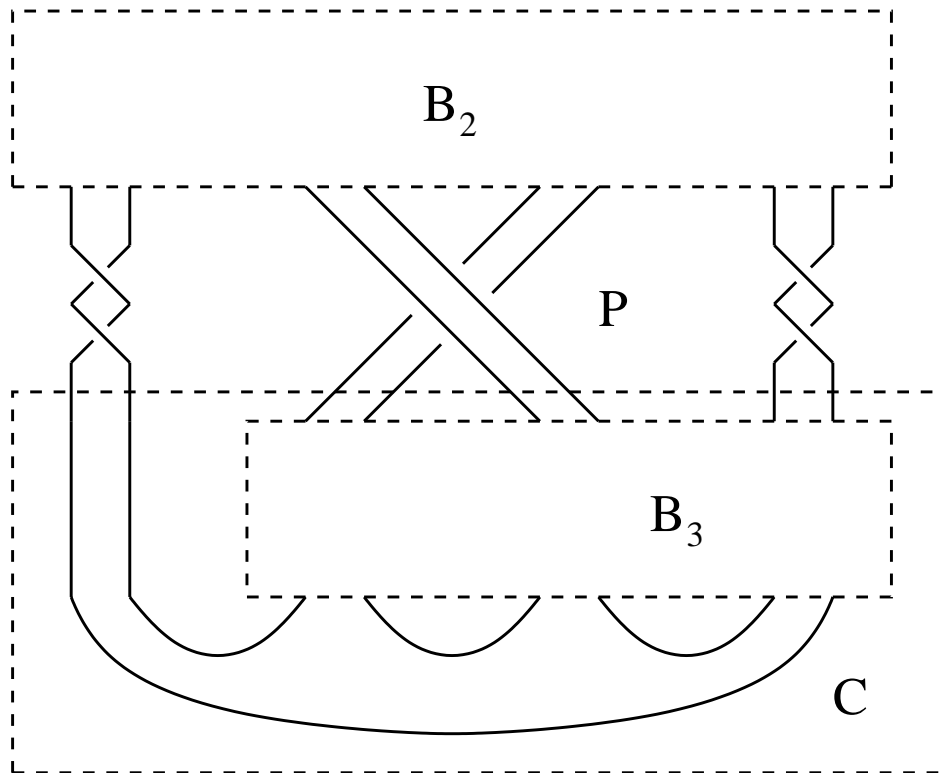


Take a **bridge presentation** of the Seifert surface  $F$ .



Replace the trivial band tangles  $I_2$  and  $I_3$  by  $B_2$  and  $B_3$ , respectively, to get  $F'$ .

$F'$  is concordant to  $F$ .



The part of  $F'$  inside the 3-ball  $C$  has exterior homeomorphic to the exterior of  $B_3$ .

The part of  $F'$  between the two tangle balls has exterior a product, so the exterior of  $F'$  is excellent by the Gluing Lemma.

A second application of the Gluing Lemma shows that  $K'$  is excellent, hence hyperbolic.

## PUMPING UP THE VOLUME

Let  $M$  be a compact 3-manifold. The **Haken number** of  $M$  is the maximal number of disjoint, incompressible,  $\partial$ -incompressible, pairwise non-parallel surfaces that can be embedded in  $M$ .

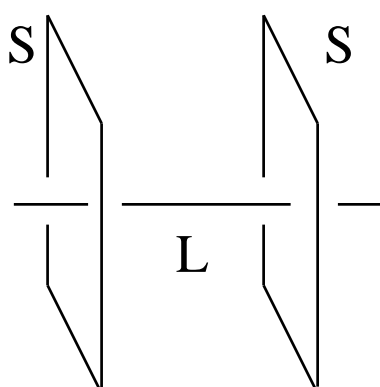
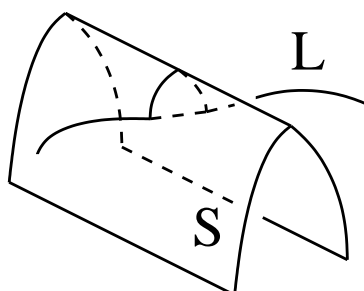
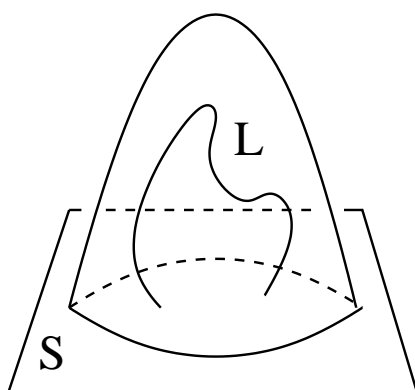
Replace  $B_n$  by a sum of many copies of  $B_n$  to get the Haken number of  $S^3 - N(K')$  arbitrarily large.

**Observation:** Let  $\mathcal{M}$  be a set of hyperbolic 3-manifolds whose Haken numbers are unbounded. Then their volumes are unbounded.

Suppose the volumes are bounded by  $C$ . By Jorgensen's Theorem there are finitely many hyperbolic 3-manifolds  $N_i$  such that each  $M \in \mathcal{M}$  can be obtained by Dehn filling on some  $N_i$ .

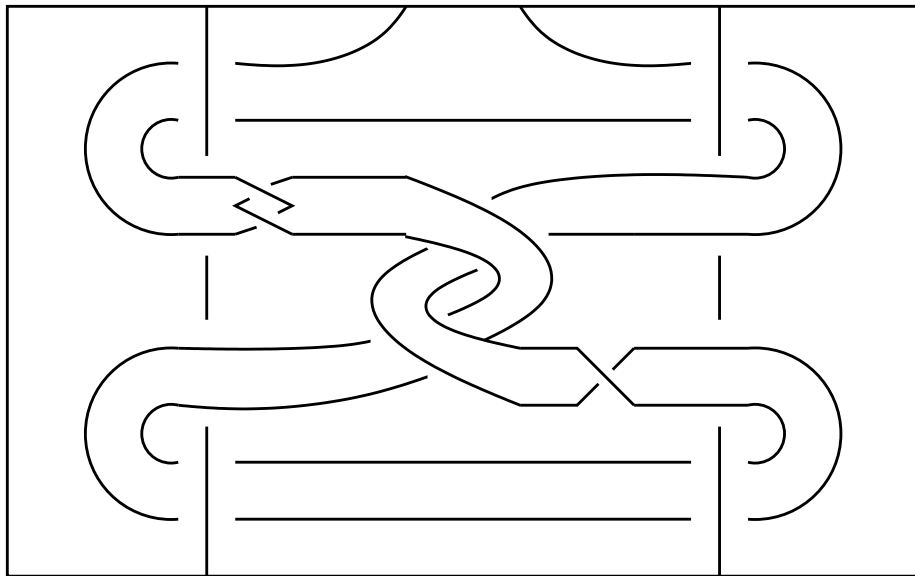
Choose an  $M$  whose Haken number is larger than those of all the  $N_i$ . Choose a 2-manifold  $S$  in  $M$  realizing the Haken number.

Put the union  $L$  of the cores of the filled in solid tori in minimal general position with respect to  $S$ . Then  $S \cap N_i$  violates the Haken number for  $N_i$ .



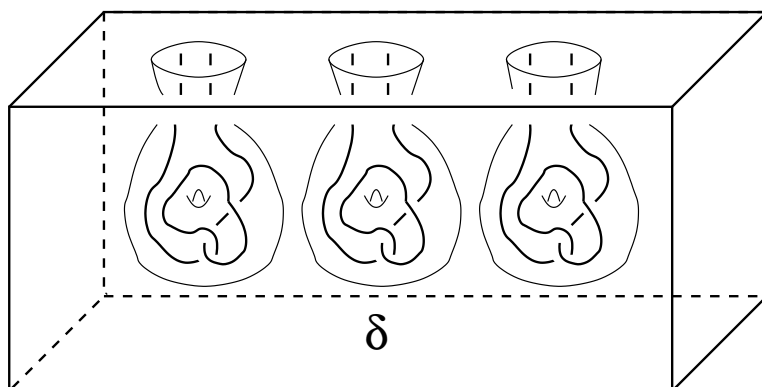
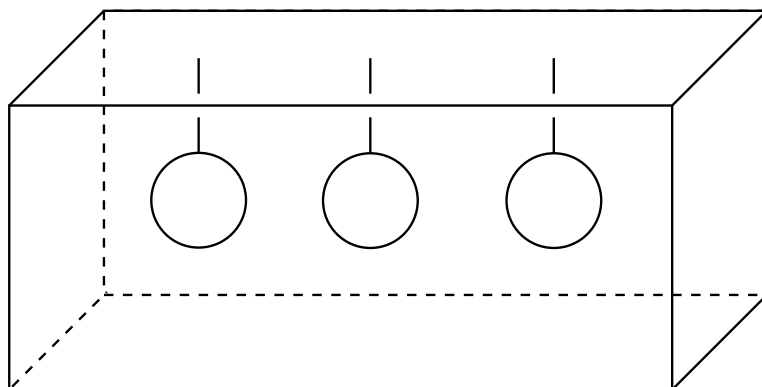
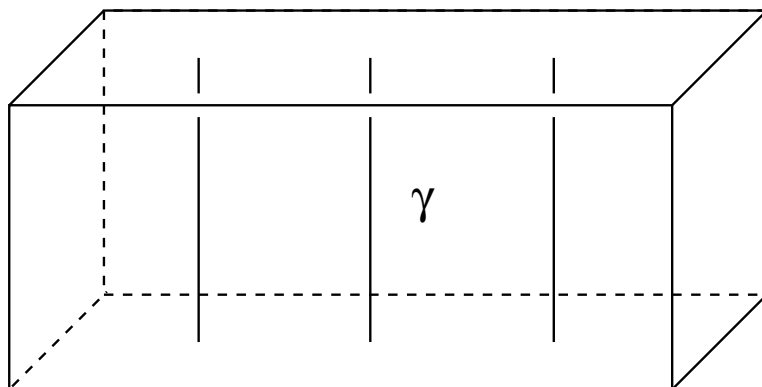
## MAKING A MAP

An  $n$ -tangle  $(B, \alpha_1 \cup \dots \cup \alpha_n)$  is a **boundary  $n$ -tangle** if there are disjoint arcs  $\beta_1, \dots, \beta_n$  in  $\partial B$  with  $\beta_i \cap (\alpha_1 \cup \dots \cup \alpha_n) = \partial\beta_i = \partial\alpha_i$  such that the  $\alpha_i \cup \beta_i$  bound disjoint surfaces in  $B$ .



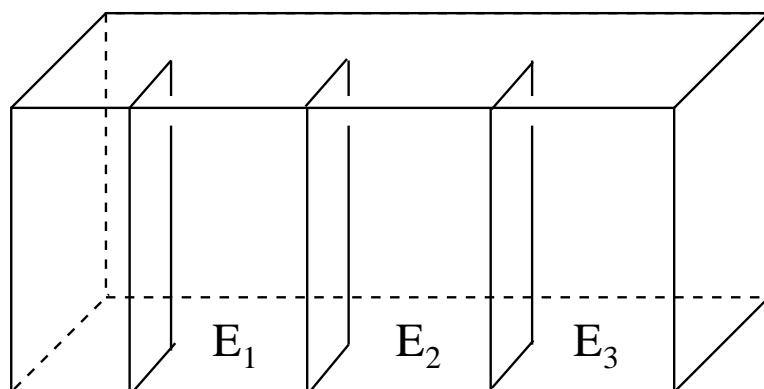
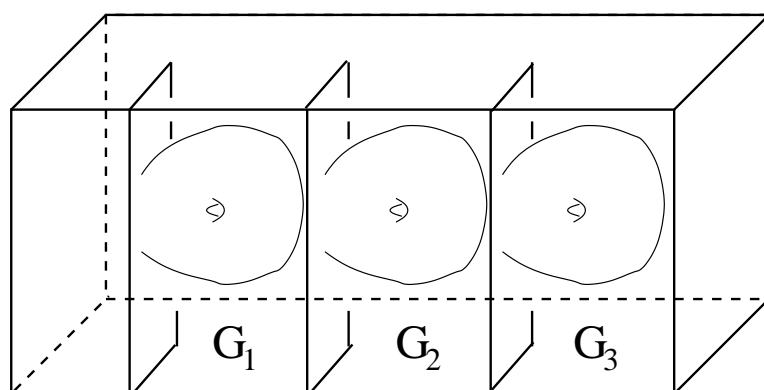
Any excellent  $n$ -tangle  $(B, \gamma)$  can be turned into an excellent boundary  $n$ -tangle  $(B, \delta)$ .

Schematic:



Let  $(B, \varepsilon)$  be the trivial  $n$ -tangle. There is a map  $g : (B, \delta) \rightarrow (B, \varepsilon)$  with  $g|_{\partial B} = id_{\partial B}$ .

Crush the surface  $G_i$  bounded by  $\delta_i \cup \beta_i$  to a disk  $E_i$  bounded by  $\varepsilon_i \cup \beta_i$ . Then extend to  $B$ .



This yields a map on the exteriors of the tangles. Extend by the identity to get  $f : (S^3, K') \rightarrow (S^3, K)$ .

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**[M. 83]** R. Myers, *Homology cobordisms, link concordances, and hyperbolic 3-manifolds*, Trans. Amer. Math. Soc. Vol. 278, No. 1, (1983), 271-288.

**[Silver-Whitten 05]** D. Silver and W. Whitten, *Hyperbolic covering knots*, Alg. and Geom. Top., Vol. 5, (2005), 1451-1469.

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