CHAPTER 4

Parametrizations of Normal and Almost Normal Surfaces

In this Chapter we present two quite useful features of normal (and almost normal) surfaces. One is the parametrization of normal isotopy classes of normal surfaces by n–tuples of nonnegative integers. This provides the description of normal and almost normal surfaces with notation useful for computation and decision problems. Secondly, the parameterizations have the added feature of connecting standard cut-and-paste techniques from 3–manifold topology (geometric addition) to the vector algebra of $\mathbb{R}^n$. We give three of the most used methods of parametrization: normal solution space, quadrilateral solution space and edge-weight solution space. All of these representation have similar features but offer computational alternatives.

4.1. Normal Solution Space

Suppose $\mathcal{T}$ is a triangulation of the 3–manifold M. Choose some ordering $\{d_1, \ldots, d_n\}$ of the normal triangle and normal quad types for $\mathcal{T}$; $n = 7t$, where t is the number of tetrahedra of $\mathcal{T}$. Suppose F is a normal surface in M (with respect to $\mathcal{T}$). The surface F meets each tetrahedron of $\mathcal{T}$ in components that lift to normal triangles and normal quads; we associate with F the $7t$-tuple of nonnegative integers in $\mathbb{R}^{7t}$

$$F \rightarrow (z_1, z_2, \ldots, z_{7t}),$$

where z_i is the number of copies of the elementary disk type d_i in the lift of F. Actually, the association is between normal isotopy classes and nonnegative integer $7t$–tuples; however, we will assume this is understood and not necessarily distinguish between a normal surface and its normal isotopy class. In Figure **??** (f-normal-parameter) we give two examples. Part A is the one-tetrahedron triangulation $\mathcal{T}$ of the solid torus $\mathbb{T}$ (minimal $\{3, 2, 1\}$–layered triangulation of the solid torus) along with the parametrization of some normal surfaces. Part B is the one-tetrahedron, two-vertex triangulation of S^3. In each of these examples, there are 7 elementary disk types, four normal triangles and three normal quadrilaterals. Thus there is a parametrization of the normal surfaces in both of these examples with certain 7–tuples of nonnegative integers in $\mathbb{R}^7$. Part C is a three-tetrahedron (two-vertex) triangulation of the solid torus along with the parametrization of some normal surfaces (see Exercise **??**(from chapter 2)). Here the parametrization is to $\mathbb{R}^{21}$.

Each normal surface can be associated with a unique integer lattice point in the positive orthant of $\mathbb{R}^{7t}$; however, if we allow singular normal surfaces, the correspondence can be many-to-one. In Figure **??** (f-normal-parameter), some of the examples give multiple representations to the same n–tuple (see below). Furthermore, not every nonnegative integer lattice point in $\mathbb{R}^{7t}$ corresponds to a normal surface in M (with respect to the triangulation $\mathcal{T}$). There is a constraint on an integer lattice point in $\mathbb{R}^{7t}$ for it to correspond to a normal surface in M and an additional constraint so that it corresponds to a unique embedded normal surface. We now consider these constraints.

4.1.1. Matching equations. The first constraint is a matching constraint and is associated with matching the elementary disk types in one tetrahedron to the elementary disk types in another tetrahedron, where the tetrahedra have identified faces. This can be overcome by requiring that a $7t$–tuple also satisfy a system of linear equations. Each face of a tetrahedron has three normal arc types; each normal arc type is in the boundary of two elementary disk types in the tetrahedra, one normal triangle and one normal quadrilateral. If F is a normal surface in M, $\tilde{\Delta}_i$ and $\tilde{\Delta}_j$ are tetrahedra in $\mathcal{T}$ with faces $\tilde{\sigma} \subset \tilde{\Delta}_i$ and $\tilde{\sigma}' \subset \tilde{\Delta}_j$ identified, and $\alpha \subset \tilde{\sigma}$ and $\alpha' \subset \tilde{\sigma}'$ are identified arc types, we must have the same number of elementary disk types in $\tilde{\Delta}_i$ meeting $\tilde{\sigma}$ in α as we have elementary disk types in $\tilde{\Delta}_j$ meeting $\tilde{\sigma}'$ in α'. This is accomplished by requiring that a $7t$–tuple satisfy the system of linear equations

$$x_p + y_q = x_r + y_s, \tag{4.1}$$

where x_p and y_q are the number of normal triangles and normal quads in $\tilde{\Delta}_i$ that contain $\alpha \subset \tilde{\sigma} \subset \tilde{\Delta}_i$ and x_r and y_s are the number of normal triangles and normal quads in $\tilde{\Delta}_j$ that contain $\alpha' \subset \tilde{\sigma}' \subset \tilde{\Delta}_j$. See Figure (f-matching-equations). There is a matching equation for each pair of identified arc types in the faces of the tetrahedra in $\mathcal{T}$; hence, there are $6t$ such equations when M is a closed 3–manifold and $6t - \frac{3}{2}f_\partial$ in the case $\partial M \neq \emptyset$, where f_∂ is the number of faces of $\mathcal{T}$ in ∂M. We call these the *matching equations* for $\mathcal{T}$. They are completely determined by $\mathcal{T}$ and can be written from the face identifications. In Figure (f-example-match-eqs), we give the matching equations for the examples given in Part A and B of Figure **??**(f-normal-parameter). These demonstrate the number of matching equations in the cases of a bounded and a closed 3–manifold; however, they also show that the equations are not independent. Below, we show that there are e_I dependent equations, where e_I is the number of edges of the triangulation $\mathcal{T}$ in the interior of the 3–manifold M ($e_I = e$, the number of edges when M is closed). Notice that the parameterizations given in Figure **??**(f-normal-parameter) satisfy the normal equations for the corresponding examples in Figure **??**. See Exercise 4.1.5 for more examples.

If we add the conditions that $x_p \geq 0, y_q \geq 0, \forall p, q$, then there is a cone in the positive orthant of $\mathbb{R}^{7t}$, denoted $\mathcal{S}(M, \mathcal{T})$, all points of which satisfy the matching equations. We call $\mathcal{S}(M, \mathcal{T})$ the *(normal) solution cone* for $(M, \mathcal{T})$.

Note, in general, we use the convention of ordering the elementary disk types by first listing the $4t$ normal triangle types and then listing the $3t$ normal quad types. Following this we use the variables x_p for the number of normal triangles of type d_p and y_q for the number of normal quads of type d_q.

4.1.2. Quadrilateral conditions. If the integer lattice point $(x_1, \dots, x_{4t}, y_1, \dots, y_{3t})$ is in $\mathcal{S}(M, \mathcal{T})$, there is a mapping of a surface into M so that its image meets each tetrahedron in such a way that the lifts are collections of elementary disks; however, there are many possible such surfaces for the same $7t$–tuple and some may not even be realized by immersed surfaces. In Figure **??**(f-singular-normal-parameter) Part (B), we have the solution $(0,0,0,0,0,1,1)$ for the one-tetrahedron (two-vertex) triangulation of S^3. There is no way to make the two quads intersect so that we can match the edges of the quads with the corresponding face identifications unless we have the quads meet as shown. In this case we get a singular (immersed) mapping of the projective plane into S^3. Similarly, we have the same triangulation of S^3 and the solution $(0,0,0,0,2,0,0)$. Here we have two possibilities for this solution: two embedded tori or a singular (immersed) torus; these are distinct normal surfaces having the same representation into $\mathbb{R}^7$. Finally, in Part (A) we have an immersion of an annulus, which is normally homotopic to an embedding; in this case, the singular surface results from just badly placing the elementary disks in the tetrahedron.

So, in most situations we want to add a second constraint; namely, we want at most one of the quadrilateral types in a tetrahedron to have nonzero entries. If a normal surface has the property that for any tetrahedron in the triangulation, the surface meets the tetrahedron so that the lift of the surface has at most one quadrilateral type in the lift of the tetrahedron, we say the normal surface satisfies a *quadrilateral condition.* There are 3^t possible quadrilateral conditions, which are determined by restricting to certain coordinate subspaces in $\mathbb{R}^{7t}$. This is done by adding a pair of equations

$$y_{i_j} = 0 = y_{i_k},$$

for each tetrahedron of $\mathcal{T}$, where $y_{i_1}, y_{i_2}, y_{i_3}$ represent the variables for the three quad types in the tetrahedron $\tilde{\Delta}_i; j \neq k; j,k \in \{1,2,3\}$. Such a quadrilateral condition adds $2t$ additional equations to the matching equations.

The points in $\mathbb{R}^{7t}$ satisfying both the normal equations and a quadrilateral condition also determine a cone in the positive orthant of $\mathbb{R}^{7t}$ which is a face of $\mathcal{S}(M,\mathcal{T})$. We will add a subscript and distinguish these cones by $\mathcal{S}_J(M,\mathcal{T})$, understanding that J runs over the 3^t possible quadrilateral conditions.

4.1.1. LEMMA. *Let $\mathcal{T}$ be a triangulation of the 3–manifold M. For each quadrilateral condition J, there is a one-one correspondence between embedded normal surfaces in M satisfying the quadrilateral condition J and the integral lattice points satisfying both the matching equations for $\mathcal{T}$ and the quadrilateral condition J.*

PROOF. If an integral solution satisfies the quadrilateral conditions, then the elementary disks in each tetrahedron can be arranged to be disjoint. Since the solution also satisfies the matching equations, after a possible normal isotopy, the face identifications match the edges of the elementary disks giving an embedded surface. ☐

It is possible that a particular solution satisfies numerous quadrilateral conditions. Also, we remark, again, that in both Part A and Part B in Figure **??**(f-normal-parameter) there are singular normal surfaces that satisfy the matching equations and a quadrilateral condition. There is, however, a unique embedded such surface.

4.1.3. Dimension of the Normal Solution Space. From the above examples, we see that it is possible to have dependencies between the matching equations.

Below, we show that, indeed, one has a dependency for each edge interior to the triangulation. So, assume, for now, that we do have as many dependencies among the matching equations as we have edges of the triangulation $\mathcal{T}$ in the interior of M. Let $e = e_I + e_\partial$ denote the number of edges in the triangulation, where e_I is the number of edges in the interior of M and e_∂ is the number of edges in ∂M. Then, for M a closed 3–manifold, we have $7t$ variables and $6t$ equations with e dependencies. Thus, dimension of $\mathcal{S}(M,\mathcal{T}) = dim(\mathcal{S}(M,\mathcal{T})) \geq 7t - 6t + e = t + e$. For M a 3–manifold with nonempty boundary, we have the $7t$ variables but we have only $6t - \frac{3}{2}f_\partial$ equations and e_I dependencies; so, $dim(\mathcal{S}(M,\mathcal{T})) \geq 7t - 6t + \frac{3}{2}f_\partial + e_I = t + e_\partial + e_I = t + e$. So, in both cases, we have that $dim(\mathcal{S}(M,\mathcal{T})) \geq t + e$.

Motivated by these observations, we consider two families of solutions to the matching equations. For each tetrahedron $\tilde{\Delta}_i$ of $\mathcal{T}$, let $d_{i_1}, d_{i_2}, d_{i_3}, d_{i_4}$ denote the four normal triangle types and let $d_{i_5}, d_{i_6}, d_{i_7}$ denote the three normal quadrilateral types in $\tilde{\Delta}_i$. For each i (for each tetrahedron), $1 \leq i \leq t$, let

$$(3Q - 4T)_i = (z_1, \ldots, z_j, \ldots, z_{7t}),$$

where

$$z_j = -1, j = i_k, k = 1,2,3,4, z_j = 1, j = i_k, k = 5,6,7, z_j = 0, j \neq i_k, 1 \leq k \leq 7.$$

If E is an edge in the triangulation $\mathcal{T}$ and $\tilde{\Delta}_i^{\tilde{\mathrm{E}}}$ is a tetrahedron of $\mathcal{T}$ containing an edge $\tilde{\mathrm{E}}$ that projects to E, then there are two normal triangle types in $\tilde{\Delta}_i^{\tilde{\mathrm{E}}}$ that meet $\tilde{\mathrm{E}}$, say $d_{i_1^{\tilde{\mathrm{E}}}}$ and $d_{i_2^{\tilde{\mathrm{E}}}}$, and a single normal quadrilateral type in $\tilde{\Delta}_i^{\tilde{\mathrm{E}}}$ that does <u>not</u> meet $\tilde{\mathrm{E}}$, say $d_{i_3^{\tilde{\mathrm{E}}}}$. Now, for each edge E and each tetrahedron $\tilde{\Delta}_i^{\tilde{\mathrm{E}}}$ that contains an edge $\tilde{\mathrm{E}}$ that projects to E (there are valence E entries in this count and if E is the image of more than one edge in $\tilde{\Delta}_i^{\tilde{\mathrm{E}}}$, then we count $D_i^{\tilde{\mathrm{E}}}$ repeatedly), let

$$(Q - 2T)_{\mathrm{E}} = (z_1, \ldots, z_j, \ldots, z_{7t}),$$

where

$$z_j = -1, j = i_k^{\tilde{\mathrm{E}}}, k = 1,2, z_j = 1, j = i_k^{\tilde{\mathrm{E}}}, k = 3, z_j = 0, j \neq i_k^{\tilde{\mathrm{E}}}, 1 \leq k \leq 3.$$

See Figure **??**(f-basis).

4.1.2. LEMMA. *If M is a compact 3–manifold and $\mathcal{T}$ is a triangulation of M having t tetrahedra and e edges, the $t+e$ vectors $(3Q-4T)_i, 1 \leq i \leq t$, and $(Q-2T)_{\mathrm{E}}$, where E runs over the e edges of $\mathcal{T}$, are a basis for the vector subspace determined by the matching equations.*

PROOF. **NEED TO ADD PROOF.** □

4.1.3. COROLLARY. *If M is a compact 3–manifold and $\mathcal{T}$ is a triangulation of M having t tetrahedra and e edges, then the dimension of $\mathcal{S}(M,\mathcal{T})$ is $t + e$.*

4.1.4. Almost Normal Surfaces. We can incorporate almost normal surfaces into the above parametrization. A lift of a tetrahedron in the triangulation $\mathcal{T}$, has three elementary octagonal disks and twenty-five elementary tubed-disks (which are not disks but annuli). So, if we add to the normal triangle and normal quad types the almost normal octagonal disk types and the almost normal tubed-disk types in each tetrahedron, then we can adjust the matching equations to include variables corresponding to the elementary octagonal disks and the elementary tubed-disks. However, since we want at most one of these exceptional pieces to appear in an almost normal surface, we set <u>all</u> <u>but</u> <u>one</u> of these additional variables equal to zero. Furthermore, the quadrilateral conditions also need to be adjusted. Namely, if for some $\tilde{\Delta}_i$ we have one of the elementary octagonal disks $o_{i_j}, j = 1, 2, 3$, then a *quadrilateral condition* must also include that all the variables corresponding to the three quadrilaterals in $\tilde{\Delta}_i$ be set equal to zero. If we have a tubed-disk in $\tilde{\Delta}_i$, then the *quadrilateral condition* must also include that any quadrilateral type in $\tilde{\Delta}_i$ that can not be made disjoint from the tubed-disk in $\tilde{\Delta}_i$ be set equal to zero.

The parametrization of almost normal surfaces allows surfaces that have multiple copies of the distinguished octagonal disk or tubed-disk; so, integer lattice points in the solution space will correspond to embedded surfaces that are neither normal nor almost normal. This does not cause a problem; one just must be cautious that an almost normal surface has precisely one of these exceptional elementary types and be cognizant that there are both normal and neither normal nor almost normal surfaces parameterized in the solution space. In Figure **??**(f-an-parameter) we give examples of parameterizations of almost normal surfaces for the examples we gave above in Figure **??**(f-normal-parameter). In Part A, we give an almost normal octagonal annulus. In Part B, we give an almost normal octagonal 2–sphere and a tubed almost normal 2–sphere.

EXERCISES:

4.1.4. Exercise. Give the coordinates in the parametrization of the normal surfaces in the triangulations given in Figures (8-10**??**-**??**) in Chapter 1.

4.1.5. Exercise. Give the matching equations for each of the triangulations in Figure **??**(f-exer-matching-eqs). What are the dimensions of the corresponding normal solution spaces?

4.1.6. Exercise. How many distinct cones do we get for the quadrilateral conditions in each of the examples in the preceding exercise?

4.1.7. Exercise. From the normal equations in Exercise 4.1.5 find the embedded, connected normal surfaces in each of these examples. Later, compare your result to what you get using the algorithm in Section **??**.

4.2. Quadrilateral Solution Space

In this section we show that the quadrilaterals in the induced cell decomposition of a normal surface completely determine the normal surface. This is then used to parameterize normal surfaces via their quadrilaterals.

4.2.1. Edge Relations. Suppose M is a compact 3–manifold and $\mathcal{T}$ is a triangulation of M. Furthermore, suppose E is an edge of $\mathcal{T}$ and E is in the interior of M. Select a direction on E (we will use a right-hand-rule for ordering the tetrahedra around E) and let E_+ and E_- denote the ends of E determined by the chosen direction on E. We call E_+ and E_- the *tip* and *base* of E, respectively.

As we cycle about the edge E, we determine an (ordered) sequence of the lifts of the tetrahedra that contain E as an edge. Say $\tilde{\Delta}_{i_1}, \dots, \tilde{\Delta}_{i_j}, \dots, \tilde{\Delta}_{i_n}$ is the ordered sequence of lifts of tetrahedra and $\tilde{\mathrm{E}}_{i_j}, 1 \leq j \leq n$, the edge in $\tilde{\Delta}_{i_j}$ corresponding to the lift of E. It is possible the $i_j = i_k$ even though $j \neq k$. See Figure **??** (f-edge-cycle) and Example **??** below. Now, for each $\tilde{\mathrm{E}}_{i_j}$ that is a lift of E, there are two normal triangles and two normal quadrilaterals in $\tilde{\Delta}_{i_j}$ that meet $\tilde{\text{E}}_{i_j}$. We will denote the variables corresponding to the two normal triangle types by x_{i_j} and $x_{i'_j}$, using x_{i_j} for the normal triangle at the base of the edge $\tilde{\text{E}}_{i_j}$ and $x_{i'_j}$ at the tip of the edge $\tilde{\text{E}}_{i_j}$. We will denote the variables corresponding to the normal quadrilateral types by y_{i_j} and $y_{i'_j}$; here we will use y_{i_j} for the normal quadrilateral at the base of the edge $\tilde{\text{E}}_{i_j}$, whose edge agrees with an edge of x_{i_j} upon entering the tetrahedron $\tilde{\Delta}_{i_j}$ in our selected cyclic ordering and $y_{i'_j}$ for the normal quadrilateral at the base of the edge $\tilde{\text{E}}_{i_j}$, whose edge agrees with the edge of x_{i_j} upon exiting the tetrahedron $\tilde{\Delta}_{i_j}$ in our select cyclic ordering. Note that at the tip of the edge $\tilde{\text{E}}_{i_j}$, the roles of the two normal quadrilaterals is exchanged; i.e., it is the quadrilateral having the variable $y_{i'_j}$ that has an edge in common with the triangle having variable $x_{i'_j}$ at the tip of the edge $\tilde{\text{E}}_{i_j}$ upon entering the tetrahedron $\tilde{\Delta}_{i_j}$ in our selected cyclic ordering and it is the quadrilateral having the variable y_{i_j} that has an edge in common with the triangle having variable $x_{i'_j}$ at the tip of the edge $\tilde{\text{E}}_{i_j}$ upon exiting the tetrahedron $\tilde{\Delta}_{i_j}$ in our selected cyclic ordering. Again, see Figure **??** (f-edge-cycle).

Using this notation, we have two matching equations for the arc types in the face of $\tilde{\Delta}_{i_j}$ at the base of $\tilde{\text{E}}_{i_j}$:

$$x_{i_{j-1}} + y_{i_{j-1}} = x_{i_j} + y_{i'_j},$$

upon entering the tetrahedron $\tilde{\Delta}_{i_j}$ in our selected cyclic ordering, and

$$x_{i_j} + y_{i_j} = x_{i_{j+1}} + y_{i'_{j+1}},$$

upon exiting the tetrahedron $\tilde{\Delta}_{i_j}$ in our selected cyclic ordering. Similarly, we have two matching equations for the arc types in the face of $\tilde{\Delta}_{i_j}$ at the tip of $\tilde{\text{E}}_{i_j}$:

$$x_{i'_{j-1}} + y_{i'_{j-1}} = x_{i'_j} + y_{i_j},$$

upon entering the tetrahedron $\tilde{\Delta}_{i_j}$ in our selected cyclic ordering, and

$$x_{i'_j} + y_{i'_j} = x_{i'_{j+1}} + y_{i_{j+1}},$$

upon exiting the tetrahedron $\tilde{\Delta}_{i_j}$ in our selected cyclic ordering. With these conventions and starting in the tetrahedron $\tilde{\Delta}_{i_1}$ containing the lift $\tilde{\text{E}}_{i_1}$ of E, we have the two systems of equations:

$$\begin{array}{lcl}
x_{i_1} + y_{i_1} = x_{i_2} + y_{i'_2} & \text{and} & x_{i'_1} + y_{i'_1} = x_{i'_2} + y_{i_2} \\
x_{i_2} + y_{i_2} = x_{i_3} + y_{i'_3} & & x_{i'_2} + y_{i'_2} = x_{i'_3} + y_{i_3} \\
\vdots & & \vdots \\
x_{i_j} + y_{i_j} = x_{i_{j+1}} + y_{i'_{j+1}} & & x_{i'_j} + y_{i'_j} = x_{i'_{j+1}} + y_{i_{j+1}} \\
\vdots & & \vdots \\
x_{i_n} + y_{i_n} = x_{i_1} + y_{i'_1} & & x_{i'_n} + y_{i'_n} = x_{i'_1} + y_{i_1},
\end{array}$$

where $x_{i_{n+1}} = x_{i_1}, y_{i'_{n+1}} = y_{i'_1}, x_{i'_{n+1}} = x_{i'_1}$ and $y_{i_{n+1}} = y_{i_1}$.

If we add the equations in each of the columns we have:

$$\Sigma_{j=1}^{n}(y_{i_j} - y_{i'_j}) = 0, \tag{4.2}$$

with the subscripts reduced modulo n. There are e such equations, where e is the number of edges of the triangulation $\mathcal{T}$ in the interior of M.

Either column of equations sum to give us the same equation. Notice that it follows from this that there are e_I redundancies among the normal matching equations, 4.1 above. This provides us an alternate method to get at the dimension of $\mathcal{S}(M,\mathcal{T})$. Namely, for M a closed 3–manifold, there are $7t$ variables, $6t$ normal matching equation with e redundancies ($e_I = e$); hence, $dim(\mathcal{S}(M,\mathcal{T})) \geq t+e$. To get the dimension $t+e$ exactly, using this method, we need to show that there are exactly e redundancies. In the case M is a bounded 3–manifold ($\partial M \neq \emptyset$), there are $7t$ variables, $6t - \frac{3}{2}f_\partial$ normal matching equations and e_I redundancies. Hence, $dim(\mathcal{S}(M,\mathcal{T})) \geq t + \frac{3}{2}f_\partial + e_I = t + e_\partial + e_I = t+e$; exactly the same as in the closed case. See Exercise **??**.

Suppose F is a normal surface

If the triangulation $\mathcal{T}$ has t tetrahedra, there are $3t$ normal quadrilateral types. From the above we have e *quadrilateral matching equations* or e Q–*matching equations* given in 4.2.

4.3. Normal Equations: The Projective Solution Space

If in addition to the matching equations, we add the equation

$$\sum_{i=1}^{7t} z_i = 1,$$

the solution set (again with $z_i \geq 0, \forall i$) is a compact, convex, linear cell in $\mathbb{R}^{7t}$. It is the intersection of the "standard" $(n-1)$–simplex with the cone $\mathcal{S}(M,\mathcal{T})$ and has one less dimension than $\mathcal{S}(M,\mathcal{T})$. We call this compact, convex, linear cell the *projective solution space* and denote it by $\mathcal{P}(M,\mathcal{T})$. We use $\mathcal{P}_J(M,\mathcal{T})$ for the intersection $\mathcal{P}(M,\mathcal{T}) \cap \mathcal{S}_J(M,\mathcal{T})$; $\mathcal{P}_J(M,\mathcal{T})$ is a face of $\mathcal{P}(M,\mathcal{T})$.

Let $\|X\|$ denote the ℓ_1–norm on $\mathbb{R}^n$ ($\|X\| = \sum_{i=1}^{n}|x_i|$). If X is a solution to the matching equations, we let $\bar{X} = X/\|X\|$ denote its projection to $\mathcal{P}(M,\mathcal{T})$, the unique point where the ray from the origin through X meets the hyperplane $\sum z_i = 1$. If X is an integer lattice points, then $\bar{X}$ is rational; conversely, every rational point in $\mathcal{P}(M,\mathcal{T})$ is the projection of an integer lattice point in the normal solution space, $\mathcal{S}(M,\mathcal{T})$. Two solutions X and X' in $\mathcal{S}(M,\mathcal{T})$ are said to be *projectively equivalent* if $\bar{X} = \bar{X}'$. If X and X' are also are both integer lattice points, then there is a unique smallest (smallest norm) integer lattice point Y and positive integers k and k' so that $X = kY$ and $X' = k'Y$. Of course, Y is projectively equivalent to both X and X'. The compact, convex, linear cell $\mathcal{P}(M,\mathcal{T})$ has a natural cell structure determined by its intersections with the coordinate planes. If X is a solution in $\mathcal{S}(M,\mathcal{T})$ and $\mathcal{C}(X)$ is the smallest (lowest dimensional) cell in $\mathcal{P}(M,\mathcal{T})$ containing $\bar{X}$, we call $\mathcal{C}(X)$ the *carrier of* X. Each compact, convex, linear cell $\mathcal{P}_J(M,\mathcal{T})$ is a face of $\mathcal{P}(M,\mathcal{T})$ and if the carrier of F is a face of $\mathcal{P}_J(M,\mathcal{T})$, then every surface that projects into the carrier of F satisfies the quadrilateral condition J. In particular, if F is an embedded normal surface, then there is a unique embedded normal surface for every nonnegative integer lattice point that projects into the carrier of F. Finally, if the integer lattice point F in $\mathcal{S}(M,\mathcal{T})$ can be written as a sum $F = X+Y$, where X and Y are also integer lattice point in $\mathcal{S}(M,\mathcal{T})$, then $\bar{X}$ and $\bar{Y}$ are in $\mathcal{C}(F)$, the carrier of F; specifically, $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are faces of $\mathcal{C}(F)$.

4.3.1. Vertex solutions. The vertices of $\mathcal{P}(M,\mathcal{T})$ are rational points (all coordinates are rational); so, for any vertex there are nonnegative integer lattice points projecting to that vertex. If V is an integer solution in $\mathcal{S}(M,\mathcal{T})$ and $\bar{V}$ is a vertex of $\mathcal{P}(M,\mathcal{T})$, we call V a *vertex solution.* A vertex solution is an integer solution that projects to a vertex in the projective solution space. For any of the quadrilateral conditions, the vertices of $\mathcal{P}_J(M,\mathcal{T})$ are vertices of $\mathcal{P}(M,\mathcal{T})$; so, the vertex solutions for $\mathcal{S}_J(M,\mathcal{T})$ are just those vertex solutions of $\mathcal{S}(M,\mathcal{T})$ that are in $\mathcal{S}_J(M,\mathcal{T})$.

We have the following very useful characterization of vertex solutions.

4.3.1. Lemma. *For any 3–manifold M and triangulation $\mathcal{T}$, an integer lattice point V in $\mathcal{S}(M,\mathcal{T})$ is a vertex solution if and only if $kV = X+Y$ implies that X and Y are projectively equivalent to V.*

Proof. If V is a vertex solution, then $\mathcal{C}(V) = \bar{V}$ is a vertex. So, if $V = X+Y$, then both X and Y project to faces of $\mathcal{C}(V) = \bar{V}$ and so are projectively equivalent to V.

To complete the proof, we will prove the negation; namely, we show that if V is not a vertex solution, then there are nonnegative integers k, n, m and nonnegative integer lattice points X and Y in $\mathcal{S}(M,\mathcal{T})$ so that $kV = nX + mY$ and neither X nor Y are projectively equivalent to V. Let $\mathcal{C}(V)$ be the carrier of V. Since we are assuming V is not a vertex-solution, the dimension of $\mathcal{C}(V) \geq 1$ and so, there are distinct rational points $\bar{X}$ and $\bar{Y}$ in $\mathcal{C}(V)$ so that $\bar{V} = \frac{p}{q}\bar{X} + \frac{q-p}{q}\bar{Y}$, where $0 < p < q$ are relatively prime integers. Thus we have $kV = nX + mY$ for $k = q\|X\|\|Y\|$, $n = p\|V\|\|Y\|$ and $m = (q-p)\|V\|\|X\|$. ☐

4.3.2. Fundamental solutions. One of the important properties of the normal solution space is that there is a finite set of nonnegative integer lattice points in the cone $\mathcal{S}(M,\mathcal{T})$ so that any nonnegative integer lattice point in $\mathcal{S}(M,\mathcal{T})$ is a linear combination of elements from this set, using nonnegative integer coefficients; in fact, there is a unique minimal such set. Following the usage by W. Haken, who introduced this to normal surface theory [], we call this Hilbert basis the fundamental solutions of the matching equations.

4.3.2. Theorem. *Suppose $\mathcal{T}$ is a triangulation of the compact 3–manifold M. There is a unique (minimal) set of nonnegative integer solutions $F_1, \ldots, F_K$ of the matching equations (for $\mathcal{T}$) so that if F is any nonnegative integer solution of the matching equations, then*

$$F = \sum_{i=1}^{K} n_i F_i,$$

where n_i is a nonnegative integer for all $i = 1, \ldots, K$.

Proof. We will say a collection of integer lattice points $\{F_1, \ldots, F_K\}$ in $\mathcal{S}(M,\mathcal{T})$ is a *spanning set* for $\mathcal{S}(M,\mathcal{T})$, if for any integer lattice point F in $\mathcal{S}(M,\mathcal{T})$, we have $F = \sum_{i=1}^{K} n_i F_i$, , where n_i is a nonnegative integer $\forall i$. We first prove that there exists such spanning sets.

Let $\bar{V}_1, \ldots, \bar{V}_n$ be the vertices of the compact, convex, linear cell $\mathcal{P}(M,\mathcal{T})$. For each $j, 1 \leq j \leq n$, there is a unique integer lattice point V_j so that every integer lattice point that projects to $\bar{V}_j$ is a nonnegative integer multiple of V_j. Let

$R = \sum_1^n \|V_j\|$. There is a finite set of integer lattice points $F_1, \dots, F_m$ in $\mathcal{S}(M, \mathcal{T})$ with $\|F_i\| \le R$. We claim that this set spans $\mathcal{S}(M, \mathcal{T})$.

If this is not the case, then there is some integer lattice point $X \in \mathcal{S}(M, \mathcal{T})$ and X can not be written $X = \sum_1^m n_i F_i$, where $n_i, 1 \le i \le m$, is a nonnegative integer. Select such an X having smallest norm among all such integer lattice points in $\mathcal{S}(M, \mathcal{T})$ that can not be written an a nonnegative integer linear combination of $F_1, \dots, F_m$. Note that since $\bar{V}_j, 1 \le j \le n$, are the vertices of the compact, convex linear cell $\mathcal{P}(M, \mathcal{T})$,

$$\bar{X} = \sum_1^n r_j \bar{V}_j,$$

where $0 \le r_j < 1$ are rational. Thus we have

$$X = \sum_1^n s_j V_j,$$

where $s_j = r_j(\|X\|/\|V_j\|)$. But since $\|X\| > R = \sum \|V_j\|$, some $s_j > 1$. There is no loss in generality to assume $s_1 > 1$. But then $X = V_1 + (s_1 - 1)V_1 + \sum_{j \ge 2}^n s_j V_j = V_1 + X'$. But then $\|X'\| < \|X\|$ and by our selection of X, we must have $X' = \sum n_i' F_i$, where each n_i' is a nonnegative integer. However, then

$$X = V_1 + \sum n_i' F_i;$$

but V_1 is an F_i and we have a contradiction to our assumption on X. Hence, the only possibility is that X can not exist and the set of integer lattice points $F_1, \dots, F_m$ in $\mathcal{S}(M, \mathcal{T})$ with $\|F_i\| \le R$ span $\mathcal{S}(M, \mathcal{T})$.

We claim there is a unique minimal (minimal under set inclusion) spanning set. For suppose both $\{F_1, \dots, F_K\}$ and $\{F_1', \dots, F_{K'}'\}$ are minimal spanning sets for $\mathcal{S}(M, \mathcal{T})$. Then we have

$$F_i = \sum_j n_j^i F_j'; \qquad F_j' = \sum_i m_i^j F_i,$$

where n_j^i is a nonnegative integer for each i and for every $j, 1 \le j \le K'$ and m_i^j is a nonnegative integer for each j and every $i, 1 \le i \le K$. If the sets $\{F_1, \dots, F_K\}$ and $\{F_1', \dots, F_{K'}'\}$ are not equal, then we can assume notation chosen so that $\{F_1, \dots, F_K\} \not\subset \{F_1', \dots, F_{K'}'\}$ and, in particular, we can assume $F_1 \notin \{F_1', \dots, F_{K'}'\}$. The above equations give

$$F_1 = \sum_j n_j^1 F_j' = \sum_j n_j^1 \sum_i m_i^j F_i = \sum_i (\sum_j n_j^1 m_i^j) F_i.$$

It follows that $\sum_j n_j^1 m_1^j = 1$ and $\sum_j n_j^1 m_i^j = 0, i \ge 2$. From $\sum_j n_j^1 m_1^j = 1$, we have that there is a j_0 so that $n_{j_0}^1 m_1^{j_0} = 1$ and $n_j^1 m_1^j = 0, j \ne j_0$. From $\sum_j n_j^1 m_i^j = 0, i \ge 2$, we have $n_j^1 m_i^j = 0, \forall j, i \ge 2$, and so, in particular, for $j = j_0$, we have

$$n_{j_0}^1 \sum_{i \ge 2} m_i^{j_0} = \sum_{i \ge 2} n_{j_0}^1 m_i^{j_0} = 0$$

and thus $n^1_{j_0} = 0$ or $\sum_{i\geq 2} m^{j_0}_i = 0$. Since $n^1_{j_0} = 1$, it follows that $\sum_{i\geq 2} m^{j_0}_i = 0$ and therefore $m^{j_0}_i = 0, i \geq 2$. This gives that

$$F'_{j_0} = \sum_i m^{j_0}_i F_i = m^{j_0}_1 F_1 + \sum_{i\geq 2} m^{j_0}_i F_i = F_1.$$

But this is a contradiction to $F_1 \neq F'_j$ for any j. □

The unique (minimal) finite set of integer lattice points $F_1, \ldots, F_K$ in $\mathcal{S}(M, \mathcal{T})$, having the property that for any lattice point F in $\mathcal{S}(M, \mathcal{T})$, $F = \sum_{i=1}^{K} n_i F_i$, where each n_i is a nonnegative integer, is called the *fundamental solutions* for $\mathcal{S}(M, \mathcal{T})$. The subcollection $F_{J_1}, \ldots, F_{J_{K_J}}$ corresponding to those fundamental solutions $F_1, \ldots, F_K$ that are also common to the cone $S_J(M, \mathcal{T})$ form the fundamental solutions for the cone $S_J(M, \mathcal{T})$.

As for vertex solutions, we have the following very useful characterization of fundamental solutions.

4.3.3. LEMMA. *For any 3–manifold M and triangulation $\mathcal{T}$, an integer lattice point F in $\mathcal{S}(M, \mathcal{T})$ is a fundamental solution if and only if $F = X + Y$ implies that $X = 0$ or $Y = 0$.*

PROOF. See Exercise **??**. □

EXERCISES:

4.4. Geometric Addition of Normal Surfaces

4.4.1. Canonical form. Normal surfaces provide an environment for the study and understanding of intersections between surfaces and, in some cases, the self intersections of a surface. In order to take advantage of this, we choose a canonical form for an embedded normal surface. If $\mathcal{T}$ is a triangulation of the 3–manifold M and F is a normal surface in M, then the lift of each component of F in a tetrahedron of $\mathcal{T}$ is an elementary disk. For an embedded normal surface, if the tetrahedron $\tilde{\Delta}_i$ projects to Δ_i, then the components of F in Δ lift to a pairwise disjoint collection of normal triangles and normal quadrilaterals in $\tilde{\Delta}_i$. The idea of a canonical form is to select the normal triangles and normal quadrilaterals nicely.

If δ is a normal triangle in the tetrahedron $\tilde{\Delta}_i$, then δ meets three edges of $\tilde{\Delta}_i$ in three distinct points and these three points determine a unique normal triangle in $\tilde{\Delta}_i$, which is the convex hull of the three points. If δ is a normal quadrilateral, then it meets the edges of $\tilde{\Delta}_i$ in four points. If these points are not coplanar (lie in the intersection of the tetrahedron with a hyperplane), then there are two "canonical" choices for a nice normal quadrilateral in $\tilde{\Delta}_i$ having the same set of vertices as δ; the difference is in our choice of a diagonal for the canonical quadrilateral, which is the same as choosing a preferred pair of vertices of δ in opposite edges of the tetrahedron. See Figure **??**. There is no reason to select one over the other; however, for each tetrahedron of $\mathcal{T}$, we make a choice (assign the preferred opposite edges) once and for all for each of the three normal quadrilateral types. A normal quadrilateral that is the union of two convex triangles is said to be *nearly convex.* A normal, convex triangle or normal, nearly convex quadrilateral in a tetrahedron, where the quadrilateral satisfies one of our preferred choices for its diagonal, is called a *canonical elementary disk* in $\tilde{\Delta}_i$. Notice that a canonical elementary disk in $\tilde{\Delta}$ is invariant under a normal isotopy that is identity on the edges of $\tilde{\Delta}$.

4.4.1. LEMMA. *A pairwise disjoint collection of elementary disks in a tetrahedron* $\tilde{\Delta}$ *is normally isotopic to a pairwise disjoint, collection of canonical elementary disks in* $\tilde{\Delta}$ *via a normal isotopy that is identity on the edges of* $\tilde{\Delta}$.

PROOF. First, observe that any one of the elementary disks in a pairwise disjoint collection of elementary disks separates $\tilde{\Delta}$ into two cells and any other elementary disk in the collection has its vertex set entirely contained in one of these cells. Next, if an elementary disk δ' and a canonical elementary disk δ in $\tilde{\Delta}$ have the same set of vertices, there is a normal isotopy of $\tilde{\Delta}$ taking δ' to δ which is identity on the edges of $\tilde{\Delta}$.

If the collection has only one elementary disk, then we are done by our second observation. Thus, we suppose the statement is true for a pairwise disjoint collection of n elementary disks in $\tilde{\Delta}$ and assume our collection has $n+1$ such disks. If one of the disks, say δ in the collection is a canonical elementary disk, then it separates $\tilde{\Delta}$, and the remaining elementary disks in our collection, into two sets having k and k' disks, where $0 \leq k, k' \leq n$. By induction we can find an appropriate normal isotopy of $\tilde{\Delta}$ taking each of these collections to a pairwise disjoint collection of canonical elementary disks; however, such a normal isotopy must be identity on δ (a canonical elementary disks acts as a barrier for such normal isotopies). Hence, we have the desired result.

So, we have the situation where we have $n+1$ elementary disks in our collection and none of them are canonical. Let δ' be any of our disks. Then there is a normal isotopy that is the identity on the edges of $\tilde{\Delta}$ taking δ' to a canonical elementary disk δ. Furthermore, the image of each of the remaining elementary disks are elementary disk and together with δ form a pairwise disjoint collection of elementary disks in $\tilde{\Delta}$. But now, this new collection has one of its members a canonical elementary disk and we can use the preceding argument to complete the induction step. □

We will say a normal surface is a *canonical normal surface* if the lifts of its components in each tetrahedron of $\mathcal{T}$ form a collection of canonical elementary disks. After the next proposition, we will typically drop the “canonical” and assume that a normal surface under consideration is canonical. Notice that there is a unique canonical normal surface having a given set of vertices in the 1–skeleton of $\mathcal{T}$, up to normal isotopy that is identity on the 1–skeleton; however, there are infinitely many that are distinct in this sense but equivalent up to normal isotopy.

4.4.2. PROPOSITION. *Suppose* $\mathcal{T}$ *is a triangulation of the* 3*–manifold* M. *Then any embedded normal surface in* M *is normally isotopic to a canonical embedded normal surface via an isotopy that is identity on the* 1*–skeleton of* $\mathcal{T}$.

PROOF. Let F be a normal surface in M (with respect to $\mathcal{T}$).

Since F is a normal surface, we have that its intersection with any tetrahedron of $\mathcal{T}$ lifts to a pairwise disjoint collection of elementary disks; and from the previous lemma, there is a normal isotopy of that tetrahedron, which is identity on its edges, taking the given set of elementary disks to a pairwise disjoint collection of canonical elementary disks. Furthermore, if we have two points on distinct edges in a face of a tetrahedron, there is a unique straight edge in the face between this pair of points. Now, this and the fact that our face identifications are affine maps, enables us to match the normal isotopies, which take the elementary disks in each tetrahedron to a collection of canonical elementary disks in that tetrahedron. So, we can construct, from the special normal isotopies on each tetrahedron, an ambient normal isotopy of M, which is identity on the 1–skeleton of $\mathcal{T}$, and takes F to a canonical normal surface. Note that to extend each of the isotopies on the various tetrahedra, it is sufficient that each of the normal isotopies on the tetrahedra be identity on the 1–skeleton. $\square$

4.4.2. intersection curves. Given two embedded normal surfaces F and F', we can assume each is a canonical normal surface and the two surfaces meet transversely. Thus the components of intersection between two embedded normal surfaces consists of simple closed curves and spanning arcs and each is a normal curve in the induced cell-decompositions of the normal surfaces. In particular, a curve of intersection between two normal surfaces consists of a union of arcs, each an intersection between a pair of elementary disks in a tetrahedron; one of the elementary disks in F and the other in F'. These intersections may come from any of the possible pairings of disks taken over the seven disk types in a tetrahedron. Pairings between distinct quadrilateral types give exceptional problems. If a curve of intersection between two normal surfaces contains an arc that is the intersection of two quadrilaterals of different type, we say that it has an *exceptional pairing* and we call the curve a *singular curve* of intersection; otherwise, we say all pairings are *regular pairings* and we call the curve a *regular curve* of intersection. Regular curves of intersection between normal surfaces have a special place in the theory and provide a direct link between the algebra of coordinate-wise addition in $\mathbb{R}^n$ and standard "cut-and-paste" techniques in low-dimensional topology. Notice that if two normal surfaces F and F' satisfy the same quadrilateral conditions, then either $F \cap F' = \emptyset$ or all curves of intersection between F and F' are regular curves of intersection.

We set up an environment to study and discuss curves of intersection between two normal surfaces. Suppose C is a curve of intersection between the normal surfaces F and F'. Then C is transverse to the 2–skeleton of $\mathcal{T}$ and thus if C meets a face of $\mathcal{T}$ it meets it in a collection of points, each the intersection between two normal arcs in that face. In Figure **??**, we show this situation and provide notation to consider C. The two intersecting segments in Part A provide a "schematic;" these segments are shown again in Part B as they might appear in a face of the triangulation $\mathcal{T}$; and in Part C, we see the origin of the schematic and the face intersections, where we have two elementary disks meeting in a tetrahedron. In Part C, we show an exceptional and a regular pairing.

For any two normal arcs in a triangle there is at least one edge of the triangle that contains an end point of both of the normal arcs. If the normal arcs intersect, the intersection point and the two end points on the same edge form a triangle in the face. A vertex angle of the two intersecting normal arcs that is interior to such a triangle is called a *fold.* Of the four vertex angles determined by the intersecting normal arcs, we have at most two that are folds and then the two folds must be opposite vertex angles. See Figure **??**. Now, suppose $\tilde{\Delta}$ is a tetrahedron and d and d' are two elementary disks intersecting in the arc a in $\tilde{\Delta}$. The elementary disks d and d' separate $\tilde{\Delta}$ into four components. If a_0 and a_1 denote the end points of a in faces $\tilde{\sigma}_0$ and $\tilde{\sigma}_1$ of $\tilde{\Delta}$, then the elementary disks d and d' meet $\tilde{\sigma}_0$ in intersecting normal arcs meeting in a_0 and meet $\tilde{\sigma}_1$ in intersecting normal arcs meeting in a_1.

The pair of intersecting normal arcs in each face determine in that face four vertex angles about each $a_i, i = 0, 1$. Each of the four vertex angles about a_i is in a unique and distinct component of the complement of $d \cup d'$ in $\tilde{\Delta}$. We will say a vertex angle at a_0 corresponds to a vertex angle at a_1 if the two vertex angles are in the same component of the complement of $d \cup d'$ in $\tilde{\Delta}$. If *mathitd* and d' are not different normal quad types, the pairing is regular, then corresponding angles are either both folds or neither are folds; furthermore, opposite vertex-angles that are not folds correspond to opposite vertex angles that are not folds. However, this is not the case for an exceptional pairing; in particular, for an exceptional pairing a vertex angle with a fold always corresponds to a vertex angle without a fold and, more importantly, opposite vertex angles that are not folds can never correspond to opposite vertex angles that are not folds. See Figure **??**.

These elementary observations lead to the following useful lemma; below they serve in defining a geometric addition for certain normal surfaces.

4.4.3. LEMMA. *Suppose $\mathcal{T}$ is a triangulation of the 3–manifold M, F and F' are normal surfaces. Each regular curve of intersection between F and F' is orientation preserving in M; hence, a regular curve of intersection between F and F' is either orientation preserving on both F and F' or orientation reversing on both.*

PROOF. Suppose C is a regular curve of intersection between F and F'. C is transverse to the 2–skeleton and can be decomposed as a collection of arcs, each of which is a component of intersection between an elementary disk in F and one in F'. As we traverse C, since C is regular, we have that opposite vertex angles on a face, as we enter a tetrahedron, correspond to opposite vertex angles on a face, as we exit that tetrahedron. Hence, as we traverse C, we preserve opposite vertex angles. It follows that C is orientation preserving (see Figure **??**).

Since a small regular neighborhood of C is a solid torus and the surfaces F and F' meet transversely, it follows that C is either orientation preserving on both F and F' or orientation reversing on both. □

4.4.3. Regular and irregular exchanges. If we have two intersecting normal arcs in a triangle, there are two ways to exchange the pairings of their end points to give two disjoint embedded spanning arcs in the triangle; but there is a unique one of these that gives a pair of disjoint normal arcs. Similarly, if two elementary disks meet in a regular pairing in a tetrahedron, there are two ways to "cut-and-paste" these disks to give two disjoint, properly embedded disks; but there is a unique one that gives a pair of disjoint elementary disks. In the case of intersecting normal arcs in a triangle, if the exchange gives disjoint normal arcs, we call the exchange a *regular exchange*; similarly, for intersecting elementary disks in a tetrahedron, if the exchange gives disjoint elementary disks, we call the exchange a *regular exchange*. Notice that for a regular pairing, the unique regular exchange for the elementary disks corresponds to the unique regular exchanges for the two pairs of intersecting normal arcs in the faces of the tetrahedron. On the other hand, there is no way to make a "cut-and-paste" along an exceptional pairing so that one has the unique regular exchanges for the two pairs of intersecting normal arcs in the faces of the tetrahedron. A "cut-and-paste" between two elementary disks meeting in a regular pairing that is not regular is called an *irregular exchange*. See Figure **??**. It is easy to see from this figure why we chose the term "fold" above.

If d and d' are two elementary disk in a tetrahedron that intersect in a regular pairing, then following a regular exchange, we have two disjoint elementary disk; however, we remain with the same disk types as d and $mathitd'$.

4.4.4. Geometric or Haken sum.

4.4.5. Schubert sum.