# Survey of Efficient Triangulations

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**Goal:** Connect triangulations to the geometry and topology of 3–manifolds.

## Strategies:

1. Understand or completely eliminate non-negative Euler characteristic normal surfaces.

2. Reduce the number of tetrahedra needed for a triangulation.

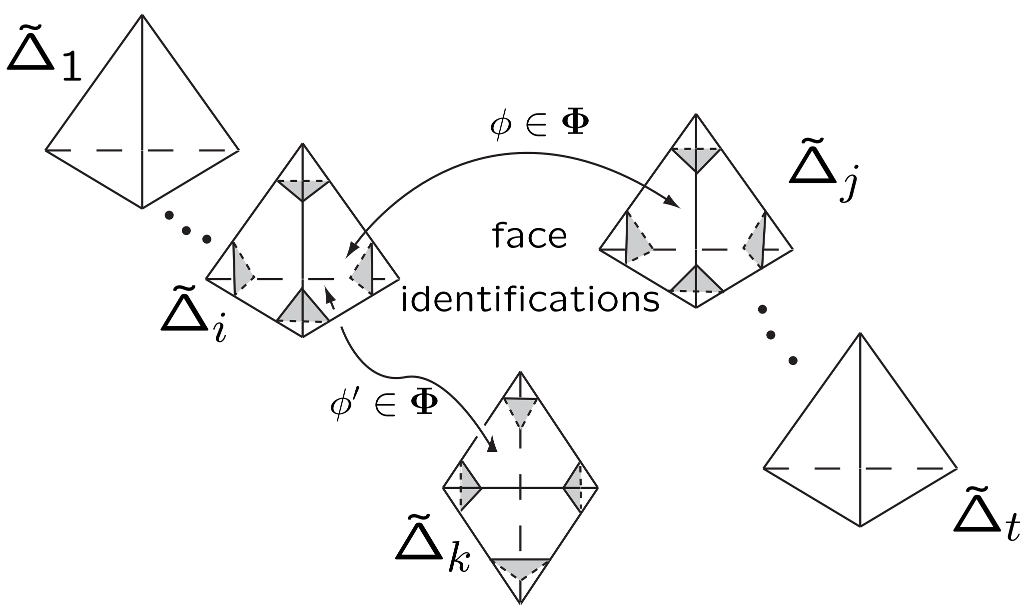
3. Connect (in a strong way) triangulations of compact 3–mainfolds having boundary to ideal triangulations of their interiors.

4. Connect (in a strong way) triangulations to useful constructions, computation of invariants, decision problems, and topological properties.

# TRIANGULATIONS

$\mathbf{\Delta} = \{\tilde{\Delta}_1, \ldots, \tilde{\Delta}_t\}$ is a pairwise-disjoint collection of oriented tetrahedra.

$\mathbf{\Phi}$ is a family of orientation-reversing, affine isomorphisms pairing faces of the tetrahedra in $\mathbf{\Delta}$




$\mathbf{\Delta}/\mathbf{\Phi}$ is a 3–complex and is a 3–manifold except possibly at the image of the vertices.

**Definition.** $\mathbf{\Delta}/\Phi = M$, a 3–manifold, then $\mathcal{T} = \{\mathbf{\Delta}, \Phi\}$ is a **triangulation** of $M$.

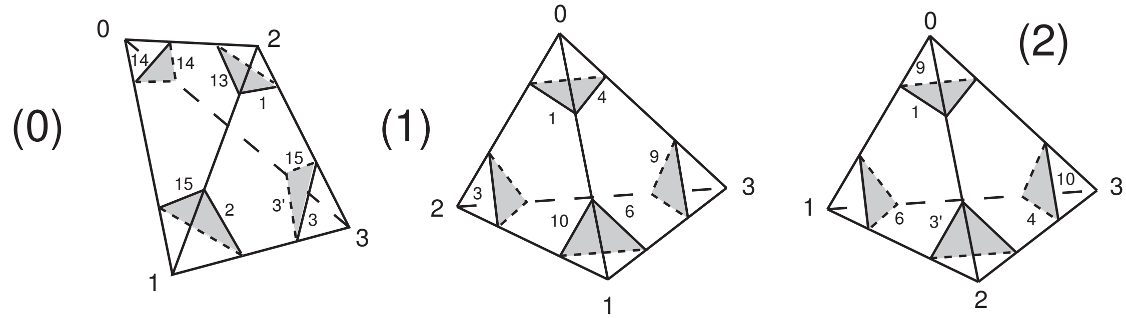
**Remarks.**

$\mathbf{1}$. Simplicies are not embedded; however, interior of simplicies are embedded.

$\mathbf{2}$. Image of tetrahedron, face, edge, vertex is called tetrahedron, face, edge, vertex.

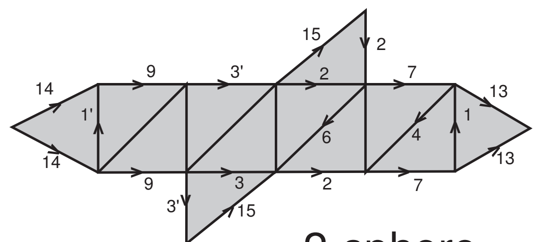
$\mathbf{3}$. $2^{nd}$–derived subdivision of a triangulation is a PL-triangulation.

# 3-Sphere



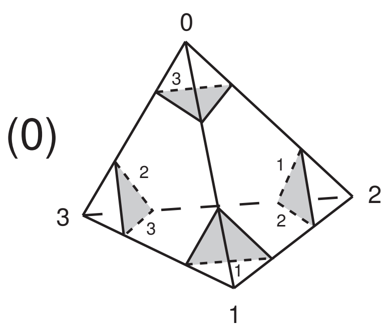

$\mathcal{T}$

| tet | 012 | 013 | 023 | 123 |
|---|---|---|---|---|
| (0) | (0)032 | (2)012 | (0)021 | (1)102 |
| (1) | (0)213 | (2)312 | (2)310 | (2)320 |
| (2) | (0)013 | (1)320 | (1)321 | (1)130 |

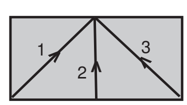


2-sphere

# solid torus




| tet | 012 | 013 | 023 | 123 |
|---|---|---|---|---|
| (0) | bddry | bddry | (0)312 | (0)230 |

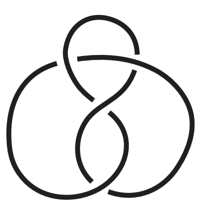


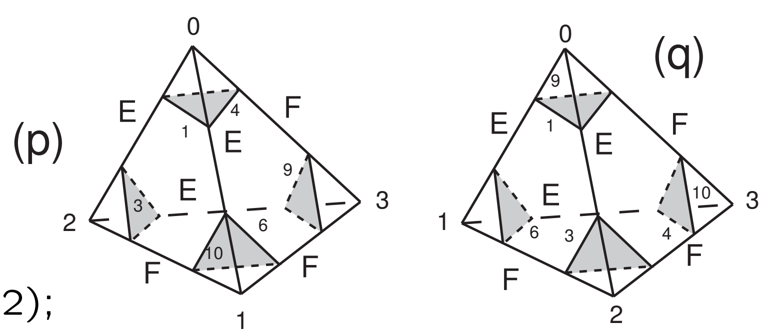
disk

**Definition.** $\mathbf{\Delta}/\Phi - \mathbf{\Delta}^{(0)} = \overset{\circ}{M}$, the interior of a compact 3–manifold, then $\mathcal{T} = \{\mathbf{\Delta}, \Phi\}$ is an **ideal triangulation** of $\overset{\circ}{M}$.

Points of $\mathbf{\Delta}^{(0)}$ are **ideal vertices**. The **index of an ideal vertex** is the genus of its vertex-linking surface.

*figure-eight* knot complement



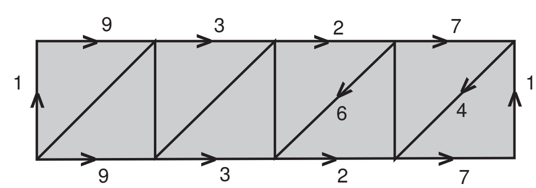



$(p)(012) \leftrightarrow (p')(012)$;

$(p)(013) \leftrightarrow (p')(312)$;

$(p)(023) \leftrightarrow (p')(310)$;

$(p)(123) \leftrightarrow (p')(320)$



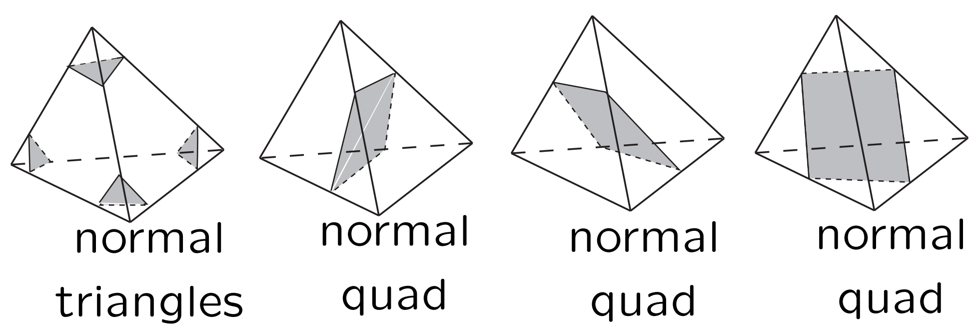
**Theorem** (Moise, Bing). *Every* 3–*manifold admits a PL-triangulation.*

**Theorem** (J-R). *Every closed* 3–*manifold admits a one-vertex triangulation.*

**Theorem** (J-R). *Every compact* 3–*manifold with nonempty boundary, no component of which is a* 2–*sphere, admits a triangulation with all vertices in its boundary and then just one-vertex in each boundary component.*

# NORMAL SURFACES

**Definition.** $\mathcal{T}$ a triangulation (ideal triangulation) of the $M$ ($\mathring{M}$), a surface in $M$ is **normal** (with respect to $\mathcal{T}$) iff it meets each tetrahedron in a collection of components that lift to normal triangles or normal quads.




**Definition.** $\mathcal{T}$ a triangulation of $M$. An isotopy of $M$ is a **normal isotopy** if it is invariant on each simplex of $\mathcal{T}$.

Two normal surfaces are **equivalent** iff there is a normal isotopy taking one to the other.

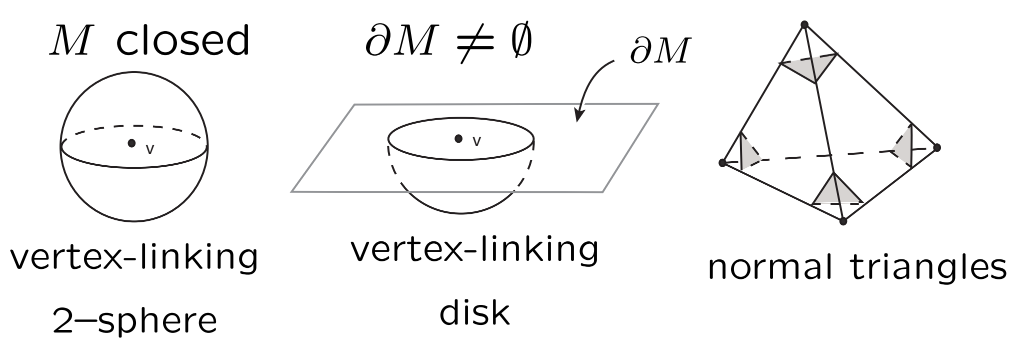
# EFFICIENT TRIANGULATIONS

## 0–efficient triangulations.

**Definition.** $M$ a compact 3–manifold, $\mathcal{T}$ a triangulation of $M$. $\mathcal{T}$ is 0–**efficient** iff

(i) ($M$ closed) the only normal 2–spheres are vertex-linking.

(ii) ($\partial M \neq \emptyset$) the only normal disks are vertex-linking.



**Proposition** (J-R). *$M$ compact 3–manifold, $\mathcal{T}$ a 0–efficient triangulation of $M$, then*

*(i) ($M$ closed) $M$ is $\mathbb{R}\mathrm{P}^2$–irreducible and $\mathcal{T}$ has only one vertex or $M = S^3$ and $\mathcal{T}$ has two-vertices.*

*(ii) ($\partial M \neq \emptyset$) $M$ is $\mathbb{R}\mathrm{P}^2$-irreducible, $\partial$–irreducible and all the vertices of $\mathcal{T}$ are in $\partial M$; furthermore, there is only one vertex in each boundary component or $M = \mathbb{B}^3$.*

**Theorem** (J-R). *$M$ irreducible and $\partial$–irreducible, then any triangulation of $M$ can be modified to a 0–efficient triangulation or it can be shown that $M$ is one of $S^3, \mathbb{R}\mathrm{P}^3, L(3,1)$ or $\mathbb{B}^3$.*

**Theorem** (J-R). *$M$ a closed, orientable 3–manifold with triangulation $\mathcal{T}$. There is an algorithm to construct a prime decomposition*

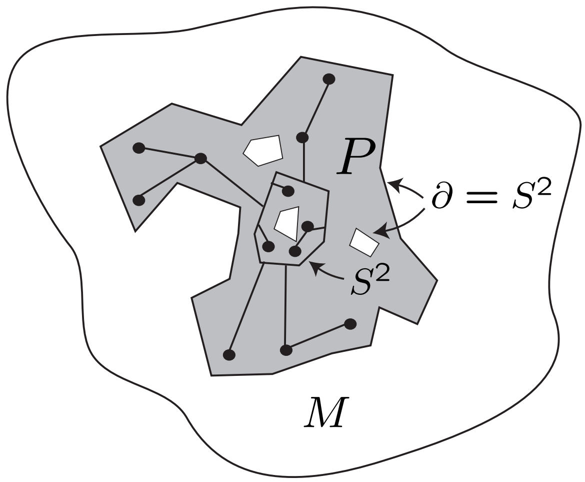
$$M = p(S^2 \times S^1)\#q(\mathbb{R}\mathsf{P}^3)\#r(L(3,1))\#M_1\# \cdots \#M_n,$$

*where $p, q$ and $r$ are nonnegative integers and each $M_i$ is given by a 0–efficient triangulation $\mathcal{T}_i, i = 1, \ldots, n$; furthermore, $\sum_{i=1}^{n} Card(\mathcal{T}_i) \leq Card(\mathcal{T})$.*

*Proof.* (OUTLINE)

**Step 1.** There is algorithm to decide if there is a non vertex-linking normal 2–sphere. If one exists, the algorithm constructs one.

**Step 2.** Engulf all vertices of $\mathcal{T}$ by a punctured 3–sphere $P$ with normal, nonvertex-linking 2–sphere boundaries - OR conclude $M = S^3$.




**Step 3.** Consider component $X$ of complement of $M \setminus \mathring{P}$. If $\partial X$ connected, then $X$ a summand in a connected sum of $M$. If $\partial X$ not connected, then $M$ must have summand $S^2 \times S^1$.

**Step 4.** If $\partial X$ not connected, can enlarge $P$ to punctured 3–sphere with some copies of $S^2 \times S^1$, (still) called $P$. Process must stop with all $\partial X$ connected (Kneser Finiteness).

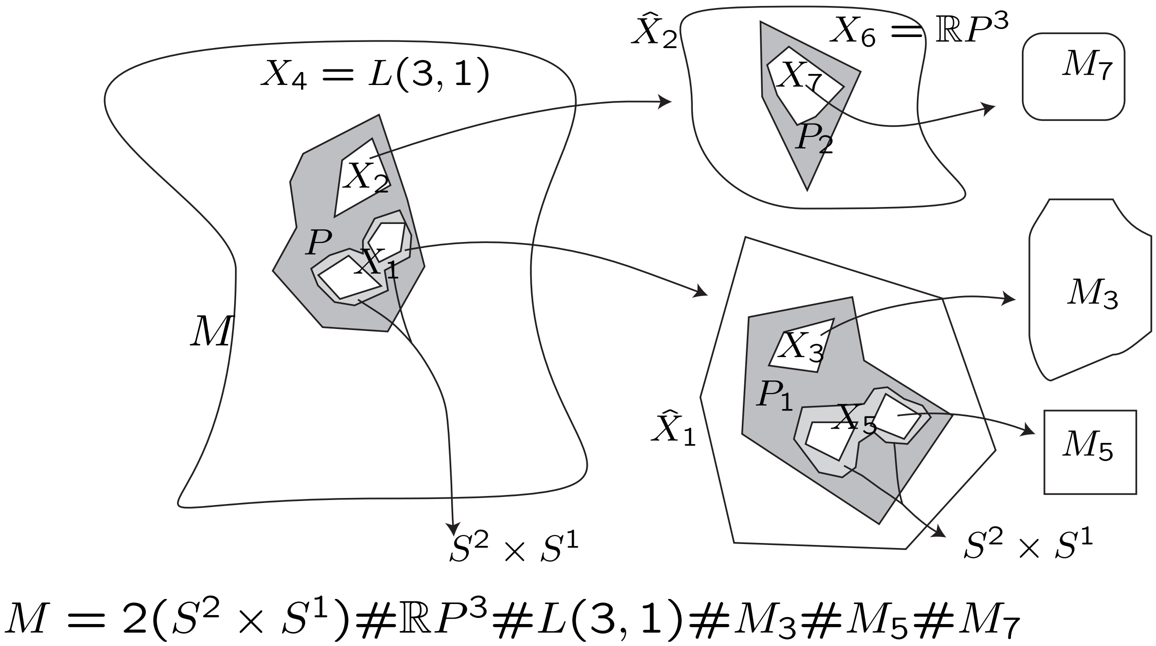
**Step 5.** For each $X$, we crush triangulation along $\partial X$. There can be obstructions:

(i) Too many product blocks, then $X = \mathbb{R}P^3$. Enlarge $P$ to a punctured 3–sphere with (possibly) $S^2 \times S^1$ and $\mathbb{R}P^3$ summands.

(ii) Too many truncated-prism blocks, then $X$ and, hence, $M$ has $L(3,1)$ as connected summand. Enlarge $P$ to a punctured 3–sphere with (possibly) $S^2 \times S^1$, (possibly) $\mathbb{R}P^3$, and $L(3,1)$ summands. Return to Step 3. Finite loop by Kneser Finiteness.

**Step 6.** We have $P$ a connected sum of a punctured 3–sphere, and possibly copies of $S^2 \times S^1$, $\mathbb{R}P^3$, and $L(3,1)$, each component $X_i$ of $M \setminus P$ has connected boundary AND there are no obstructions to crushing the triangulation $\mathcal{T}$ along the normal 2–sphere boundary of $X_i$.

Crush $\mathcal{T}$ along $\partial X_i$ for each $i$, getting a new triangulation $\mathcal{T}_i$ of $\hat{X}_i$, which is $X_i$ capped-off by a 3–cell. Each $\hat{X}_i$ is a connected summand of $M$.




**Step 7.** Start the process over at Step 1 for each $\hat{X}_i$ with triangulation $\mathcal{T}_i$. The process must stop as $Card(\mathcal{T}_i) < Card(\mathcal{T})$; in fact, $\mathcal{T}_i$ uses tetrahedra from $\mathcal{T}$ and $\sum_{i=1}^{n} Card(\mathcal{T}_i) \leq Card(\mathcal{T})$, with equality only if $\mathcal{T}$ is 0–efficient.

Hence,

$$M = p(S^2 \times S^1)\#q(\mathbb{R}\mathrm{P}^3)\#r(L(3,1))\#M_1\# \cdots \#M_n,$$

where each $M_i$ is given by a triangulation with the only normal 2–spheres vertex-linking - $\mathcal{T}_i$ is 0–efficient. Note that $L(3,1)$ has a two-tet 0–efficient triangulation. □

**Shrinking and Barrier Surfaces.** $\mathcal{T}$ a triangulation (ideal triangulation) of $M$ ($\overset{\circ}{M}$). $S$ a properly embedded surface in $M$ in general position with $\mathcal{T}^{(2)}$.

**Shrinking (Normalization):**

• The **weight** of $S$ is the cardinality of $S \cap \mathcal{T}^{(1)}$, $wt(S) = |S \cap \mathcal{T}^{(1)}|$.

• The **local Euler number** of $S$, $\lambda_\chi(S) = \Sigma_{c \neq S^2}(1 - \chi(\tilde{c}))$, $c$ runs over all non 2–sphere components of $S$ in the tetrahedra.

*Remark.* $\lambda_\chi(S) = 0$ iff each non-spherical component of $S$ in a tetrahedron lifts to a disk.

• $\sigma(S)$ is number of 0–weight curves of the intersection of $S$ with faces of $\mathcal{T}$, which are also in $\overset{\circ}{M}$.

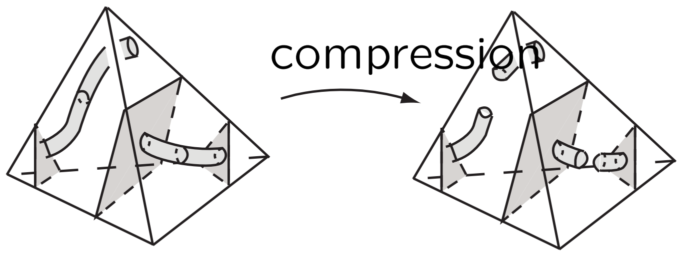
**Definition.** The **complexity of** $S$

$$C(S) = (wt(S), \lambda_\chi(S), \sigma(S)),$$

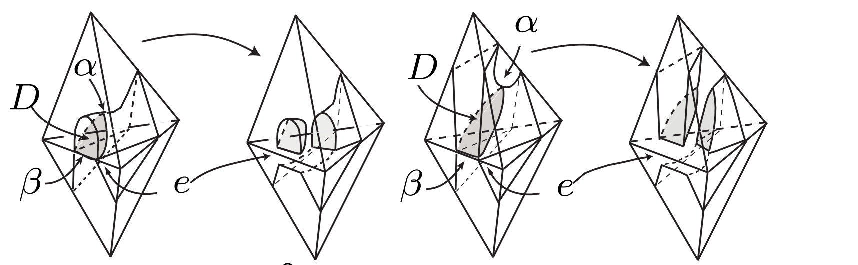
lexicographical ordered from the left.

**Normal moves.** The normal moves are:

1. *compression in interior of tetrahedron*. Reduces $\lambda_\chi(S)$, does not change $wt(S)$ or $\sigma(S)$.

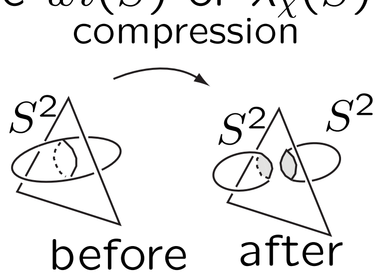


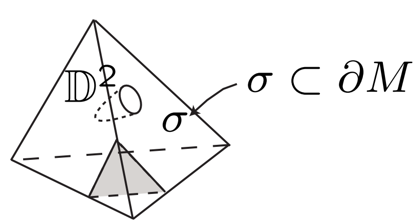

2. *isotopy or $\partial$–compression*. Reduces $wt(S)$.




isotopy, $e \subset \mathring{M}$; $\partial$–compression, $e \subset \partial M$

3. *compression in face $\subset \mathring{M}$*. Reduces $\sigma(S)$, does not change $wt(S)$ or $\lambda_\chi(S)$.



**Definition.** A finite sequence of moves 1. – 3. above on an embedded surface $S$ is called a **shrinking** of $S$.

If no moves can be made, we say $S$ is **stable**.

**Theorem** (J-R). *$S$ is a stable surface iff each component of $S$ is either normal or a properly embedded, 0-weight 2–sphere or disk in a tetrahedron.*

**EXAMPLES.**

**1**. [Haken] $M$ an irreducible and $\partial$–irreducible 3–manifold. Then for any triangulation $\mathcal{T}$ of $M$, an incompressible and $\partial$–incompressible surface is isotopy to a normal surface.

**2**.[Schubert]([Haken]) $M$ a 3–manifold. If $M$ contains an essential 2–sphere (disk), then for any triangulation of $M$ there is an an essential, normal 2–sphere (essential, normal disk).

**Barrier surfaces:** Barrier surfaces provide control over normalization.

**Definition.** $B$ is a properly embedded surface in a 3–manifold $M$. $N$ be a component of the complement of $B$. $B$ is a **barrier surface for** $N$ if any properly embedded, compact surface $F$ in $N$ can be shrunk to a stable surface in $N$.

**Proposition.** *For any triangulation, the following are barriers:*

*(i) a normal surface;*

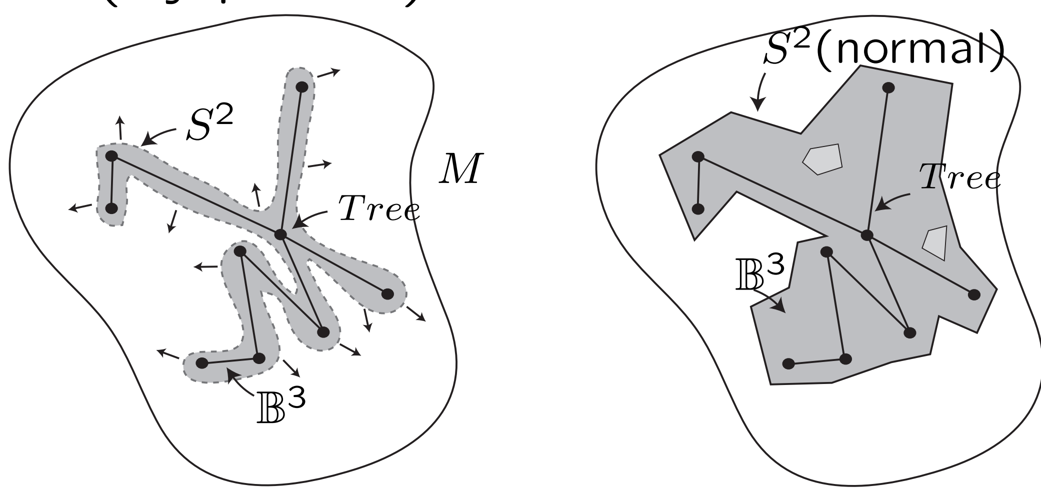
*(ii) a subcomplex;*

*(iii) a normal surface union a subcomplex of its complement;*

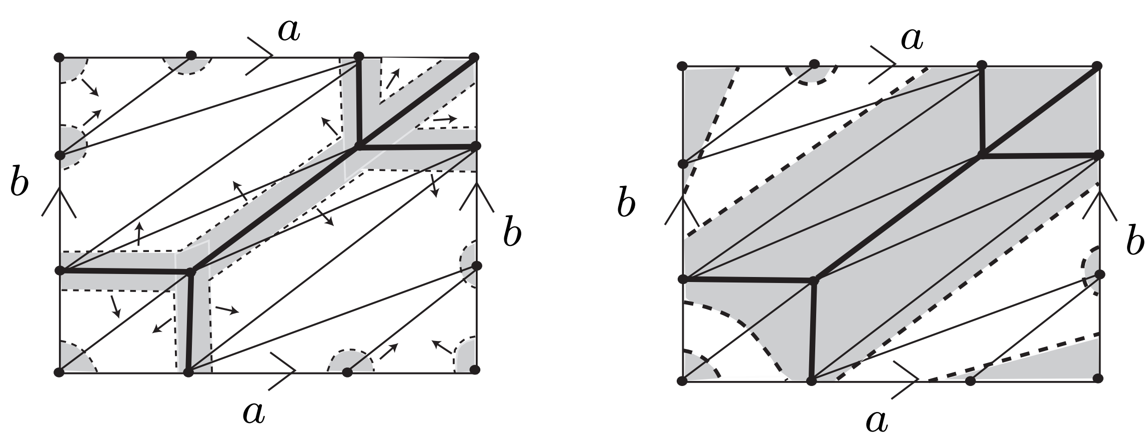
*(iv) a normal surface union a normal surface in its complement.*

**Theorem.** *$M$ a closed, irreducible 3–manifold. For any triangulation $\mathcal{T}$ of $M$, there is a normal 2–sphere bounding a 3–cell that contains all the vertices of $\mathcal{T}$ or it can be shown that $M = S^3$.*

*Proof*. (By picture)




**EXAMPLE: Engulf vertices of 2–torus**



# CRUSHING TRIANGULATIONS
(ALONG NORMAL SURFACES)

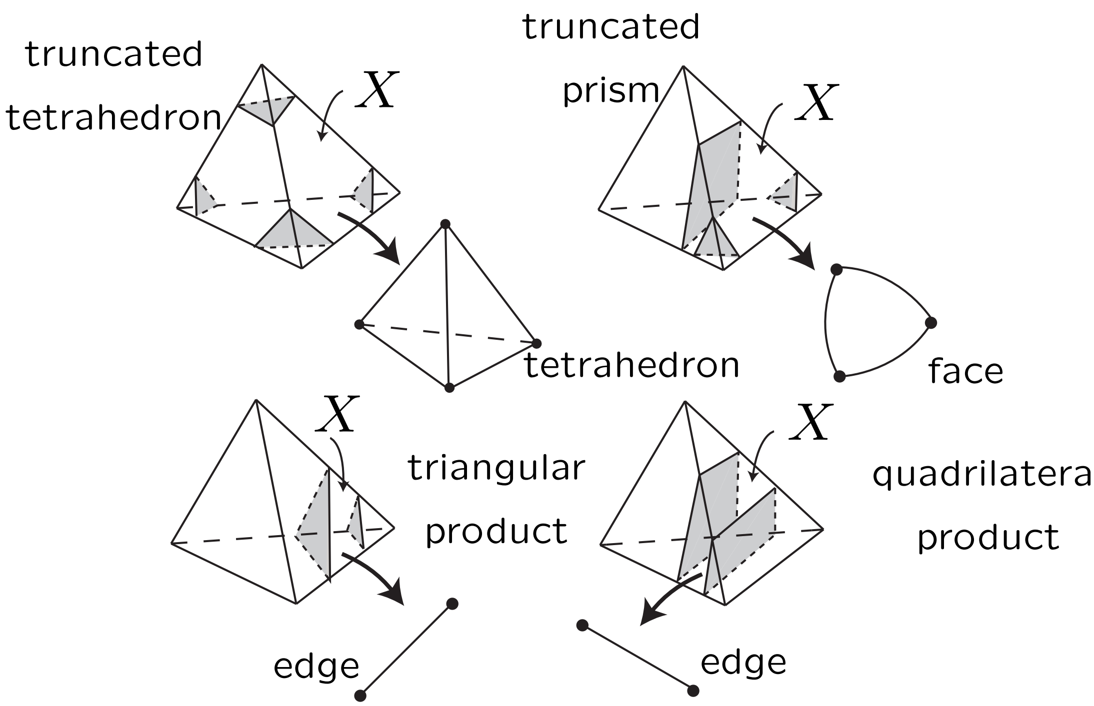
Purpose: Effective way to modify triangulations, generally, reducing number of tetrahedra and number/kinds of unwanted normal surfaces.

$\mathcal{T}$ a triangulation (or ideal triangulation) of closed 3–manifold (interior of compact 3–manifold) $M$.

$S$ normal surface embedded in $M$, $X$ closure of a component of the complement of $S$, and $X$ does not contain any vertices of $\mathcal{T}$.

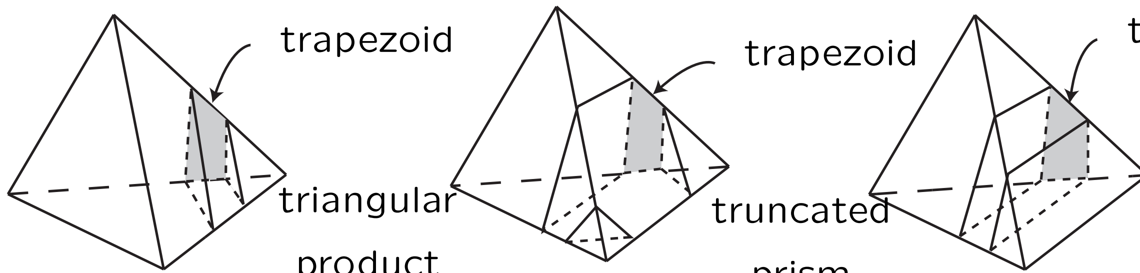
**Crushing** $\mathcal{T}$ **along** $S$ gives a particularly nice <u>ideal</u> triangulation of $\mathring{X}$.

• $\mathcal{T}$ induces a "nice" cell-decomposition on $X$, $\mathcal{C}(X)$.



• Product Regions for $X$.

**Definition.** (by picture) **trapezoids**.




**Product Region for** $X$ is $\mathbb{P}(\mathcal{C}) = \{$edges of $\mathcal{C}$ not in $S\} \cup \{$all trapezoidal faces of $\mathcal{C}\} \cup \{$triangular products in $\mathcal{C}\} \cup \{$quadrilateral products in $\mathcal{C}\}$.

Each component $\mathbb{P}_i(\mathcal{C})$ of $\mathbb{P}(\mathcal{C})$ is an $I$-bundle.

Assume:

- each $\mathbb{P}_i(\mathcal{C}) = K_i \times I$ is a trivial $I$-bundle.
- each $K_i$ is simply connected and planar.
- $\mathbb{P}(\mathcal{C}) \neq X$.

**Definition.** Under these assumptions, $X$ has a **trivial product region**.

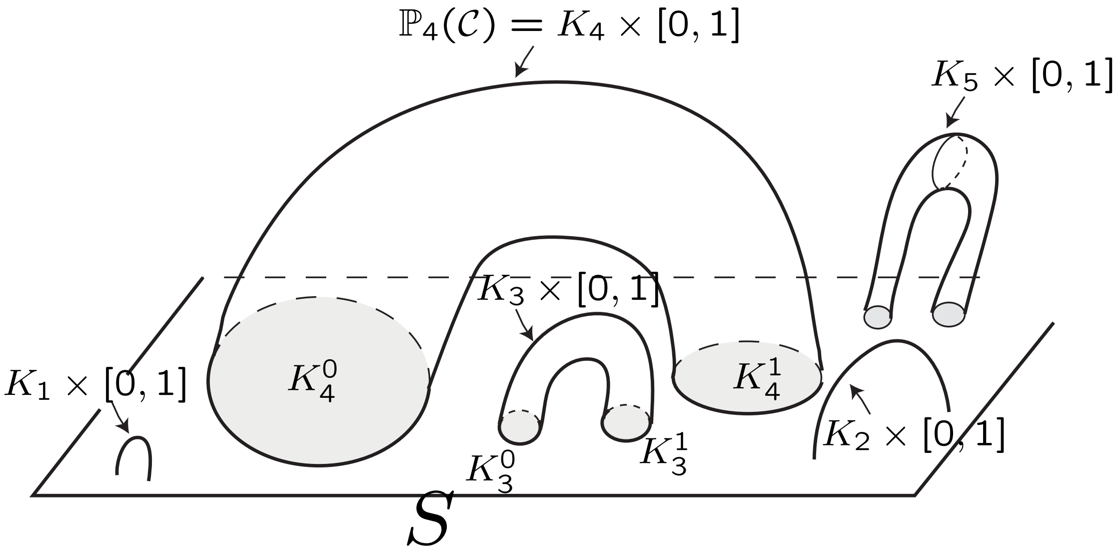
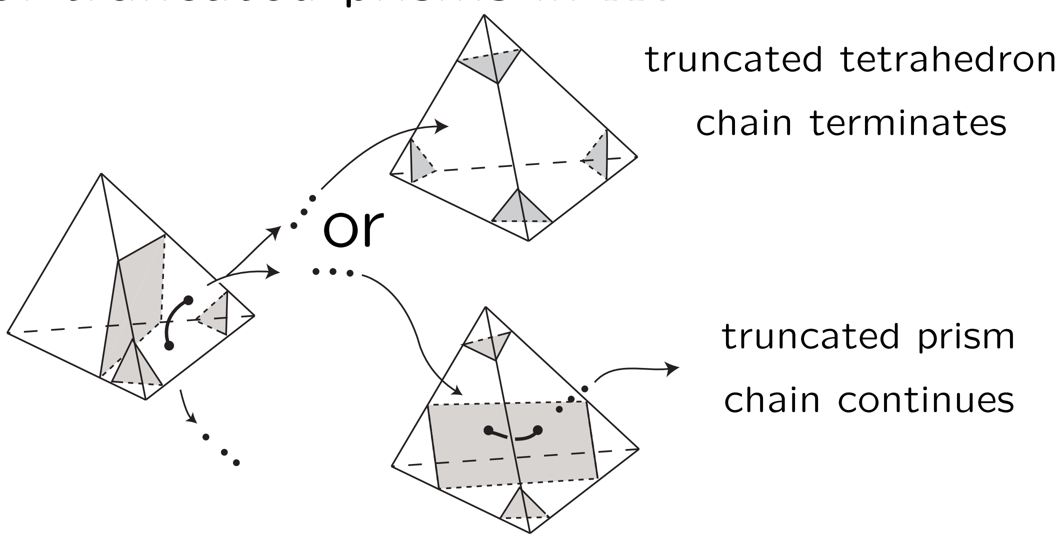



**FIGURE.** Trivial product region for $X$

- Chains of truncated-prisms in $X$.




<u>Assume</u>:

- there are no cycles of truncated-prisms not in $\mathbb{P}(X)$

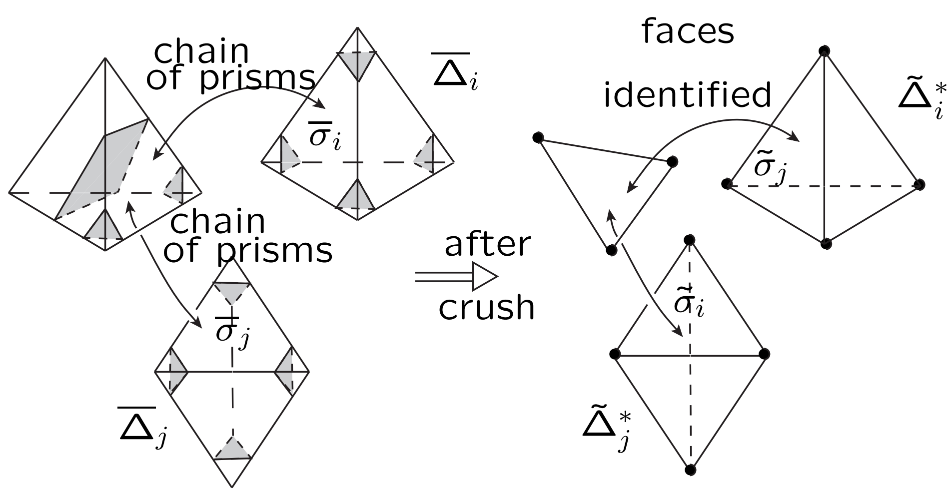
**Theorem.** *$\mathcal{T}$ a triangulation or ideal triangulation of $M$. $S$ a normal surface embedded in $M$. $X$ closure of a component of the complement of $S$ and $X$ does not contain any vertices of $\mathcal{T}$.*

*i)* $X \neq \mathbb{P}(X)$,

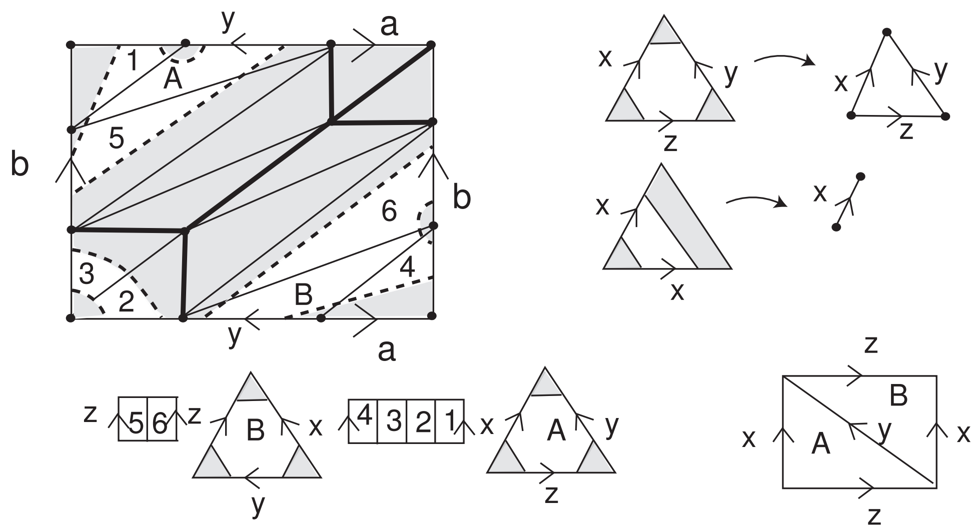
*ii)* $\mathbb{P}(X)$ *is trivial, and*

*iii) there are no cycles of truncated prisms in $X$, not in $\mathbb{P}(X)$,*

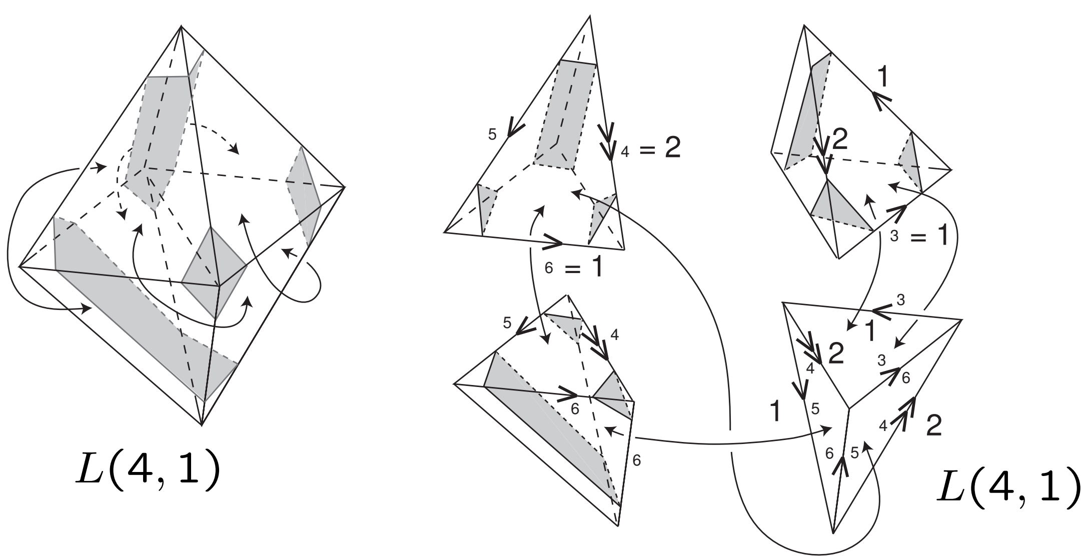
*then $\mathcal{T}$ can be crushed along $S$ to an ideal triangulation $\mathcal{T}^*$ of $\mathring{X}$.*



**Example**. Crushing along a normal curve.
Here $M = S^1 \times S^1$, $S = S^1$.
$T = S^1 \times S^1$



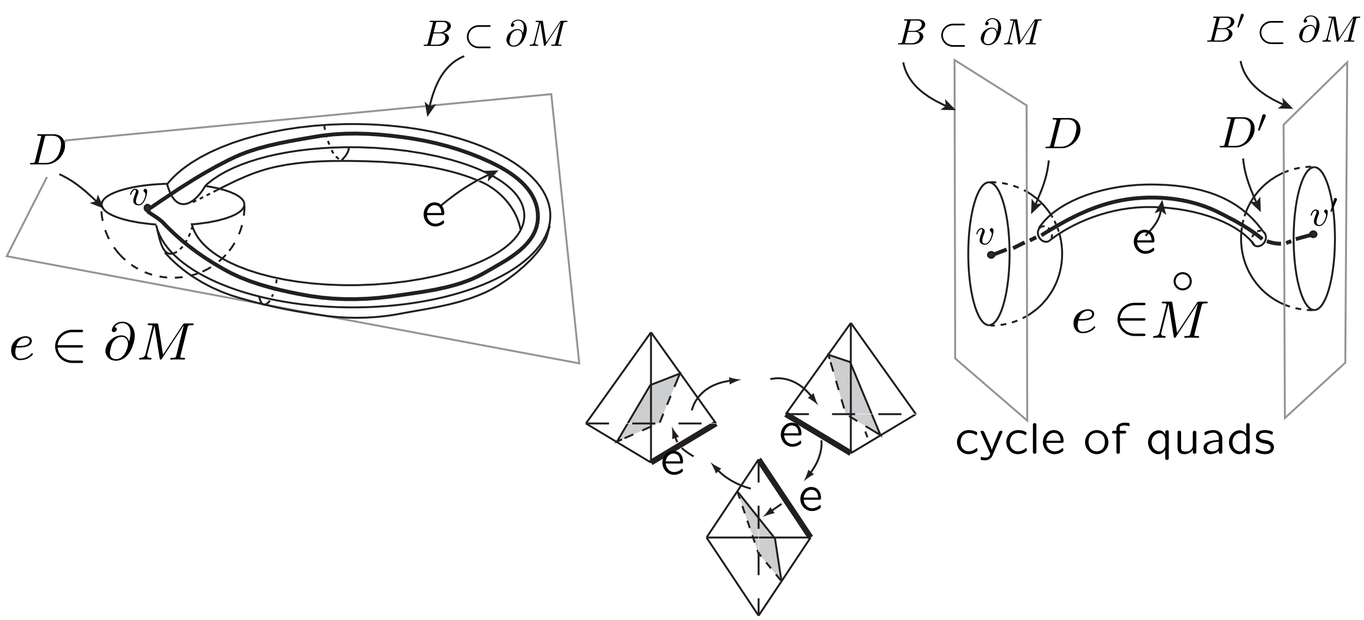

**Example**. Crushing along a 2–sphere.
Here $M = L(4,1)$, $S = S^2$.



## annular–efficient triangulations

**Definition.** Compact 3–manifold $M$ is **anannular** iff there are no properly embedded essential annuli in $M$.

**Definition.** $M$ a 3–manifold, $\mathcal{T}$ a triangulation of $M$. A normal annulus in $M$ is **thin edge-linking** iff it is normally isotopic to an arbitrarily small regular neighborhood of an edge of $\mathcal{T}$.



**Definition.** $M$ a compact 3–manifold, $\mathcal{T}$ a triangulation of $M$. $\mathcal{T}$ is an **annular–efficient triangulation** iff $\mathcal{T}$ is 0–efficient and the only normal annuli with essential boundary are thin edge-linking.

NOTE. Bachman and Schleimer suggest calling this 1/2–efficient.

**Proposition.** *$M$ a compact 3–manifold, $\partial M \neq \emptyset$, $\mathcal{T}$ an annular–efficient triangulation of $M$. Then $M$ is irreducible, $\partial$–irreducible, and anannular.*

**Theorem.** *(Bachman-Schleimer, J-R) $M$ a compact, irreducible, $\partial$–irreducible, and anannular 3–manifold with $\partial M \neq \emptyset$. If $M \neq \mathbb{B}^3$, then any triangulation of $M$ can be modified to an annular–efficient triangulation.*

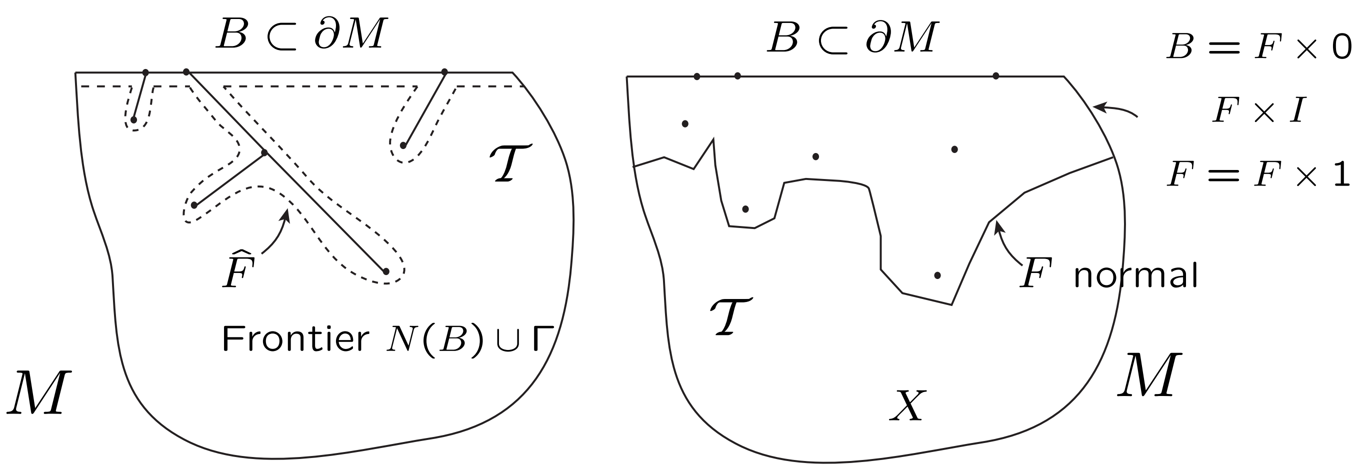
PROOF (Outline).

**Step 1.** Construct an ideal triangulation $\mathcal{T}^*$ of $\overset{\circ}{M}$.

**Theorem** (J-R). *$M \neq \mathbb{B}^3$ a compact, irreducible, $\partial$–irreducible, and anannular 3–manifold with $\partial M \neq \emptyset$, any triangulation $\mathcal{T}$ of $M$ can be modified to an ideal triangulation of $\overset{\circ}{M}$.*

*How to do this?*

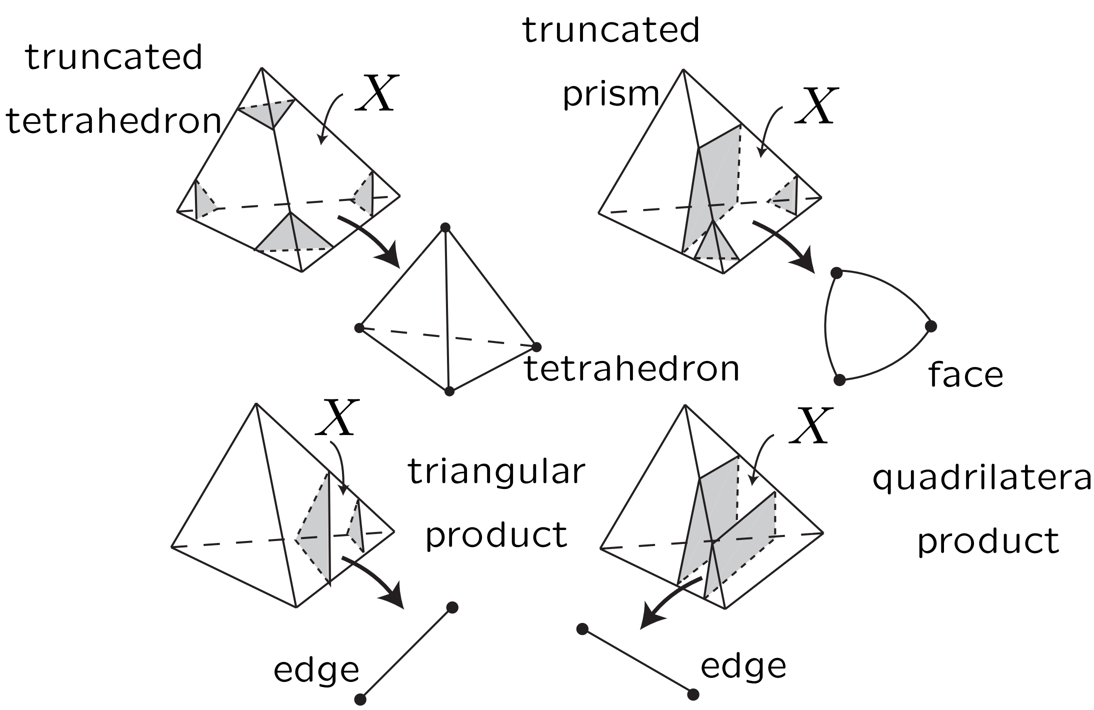
**A.** Engulf all vertices of $\mathcal{T}$ by $\partial$-parallel normal surfaces. Can do since $M$ is not an $I$–bundle.




all vertices of $\mathcal{T} \subset F \times I$

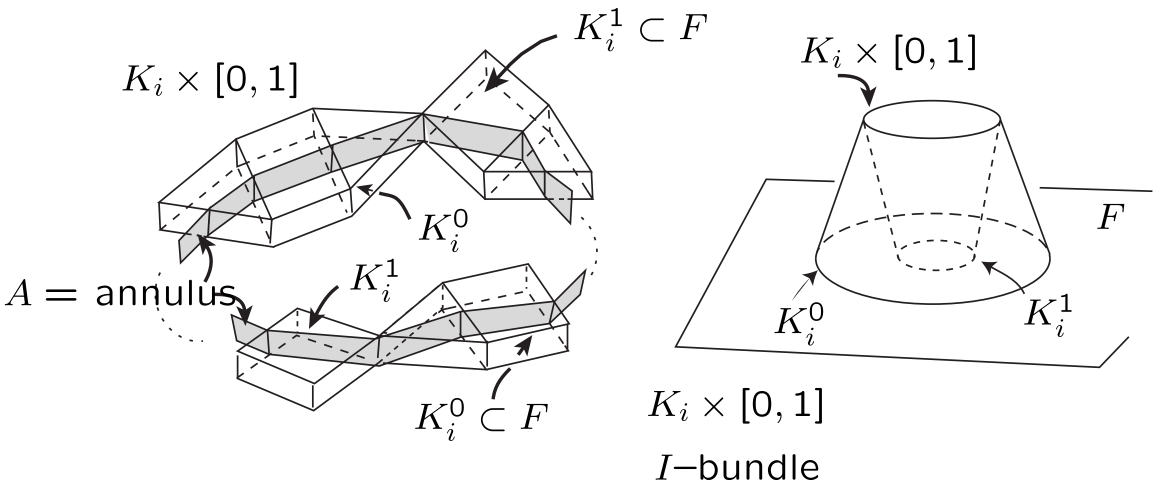
**B.** Crush the triangulation $\mathcal{T}$ along the normal surface $F$.

**Recall:** $\mathcal{T}$ induces a "nice" cell-decomposition on $X = M \setminus (F \times I)$, which is homeomorphic to $M$.




There can be obstructions to crushing.

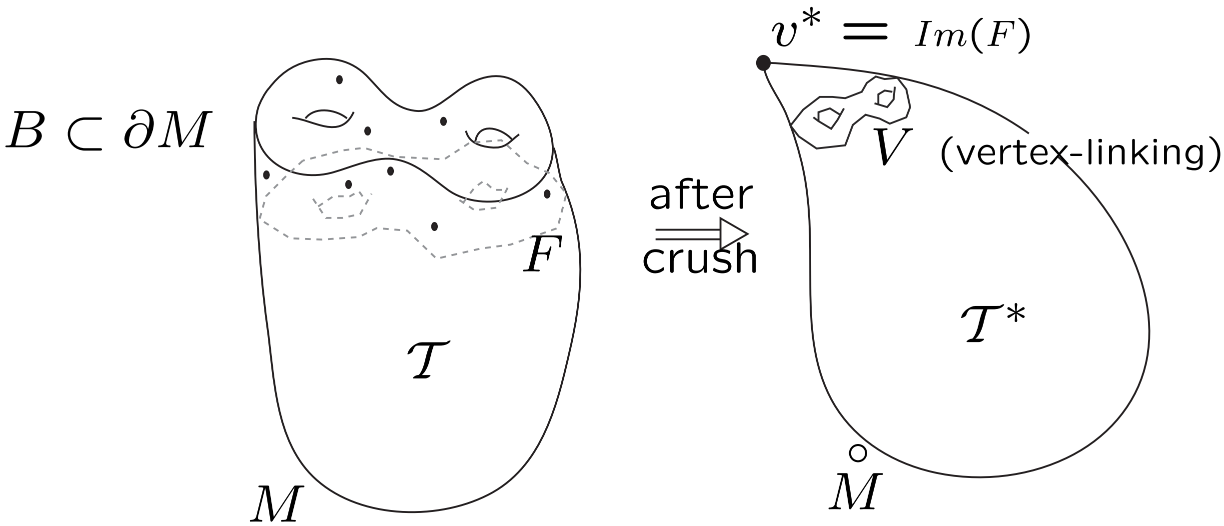
(i) Too many product blocks. The product region for $X$, $\mathbb{P}(\mathcal{C})$, must be a product $I$–bundle and is not all of $X$. However, $\mathbb{P}(\mathcal{C})$ may not be a "trivial" product region.




Must engulf $A$ by $F$. This can only happen a finite number of times by Kneser Finiteness.

(ii) Too many truncated-prism blocks. If this were the case, again we have conditions that give a contradiction to our hypothesis or we can find a new $F$. Kneser Finiteness allows this only a finite number of times.

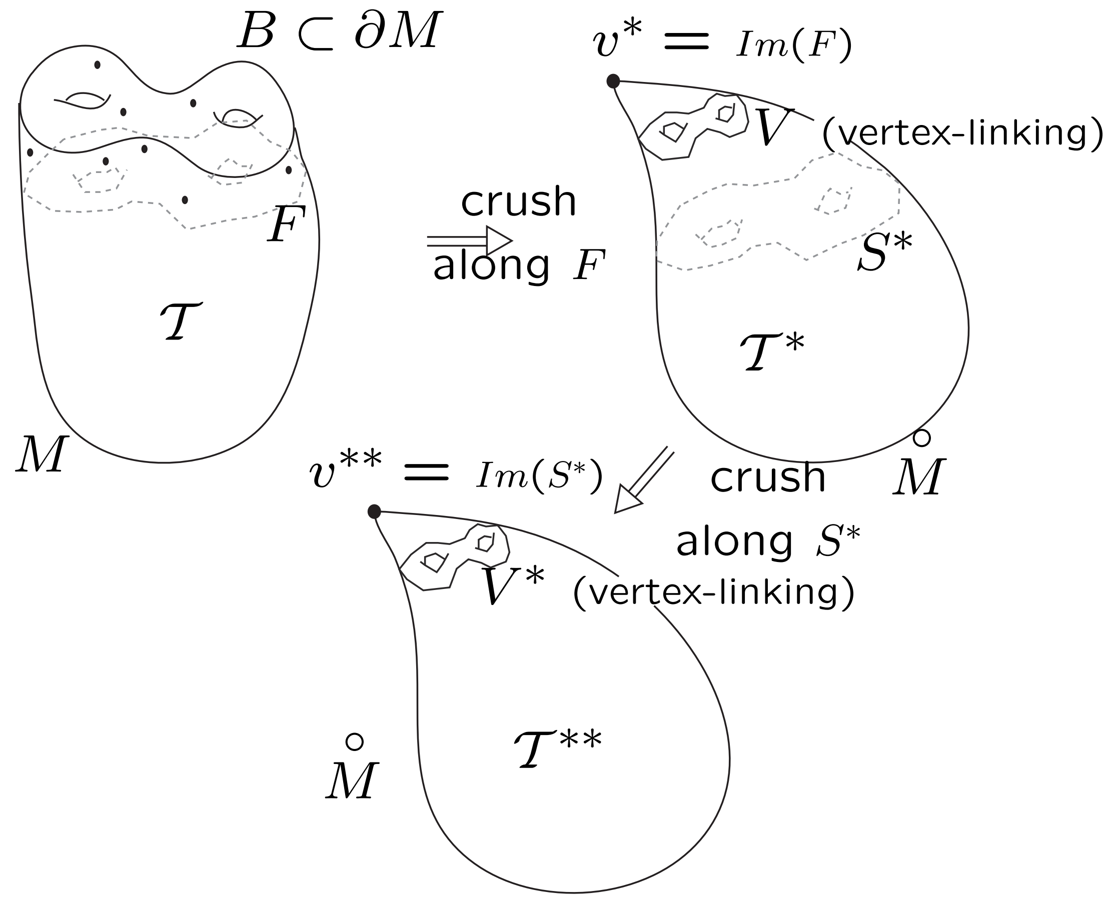
**C.** Crushing gives ideal triangulation of $\overset{\circ}{M}$.
Why do we get $\overset{\circ}{M}$?




**Step 2.** Modify ideal triangulation $\mathcal{T}^*$ to a "more" efficient ideal triangulation.

**Proposition.** *Under the hypothesis on $M$ (above) $\mathcal{T}^*$ can be modified to an ideal triangulation $\mathcal{T}^{**}$ of $\overset{\circ}{M}$ so that any normal surface isotopic to a vertex-linking surface is* <u>*normally isotopic*</u> *to that vertex-linking surface.*

NEW ideal triangulation $\mathcal{T}^{**}$ of $\overset{\circ}{M}$.




**Note:** Each time we crush to get a new ideal triangulation of $\overset{\circ}{M}$, we reduce the number of tetrahedra - every tetrahedron in $\mathcal{T}^*$ with a quadrilateral of $S^*$ is crushed and does not appear in the triangulation of $\mathcal{T}^{**}$. All other tetrahedra in $\mathcal{T}^{**}$ were tetrahedra in $\mathcal{T}^*$ (all started as tetrahedra in $\mathcal{T}$).

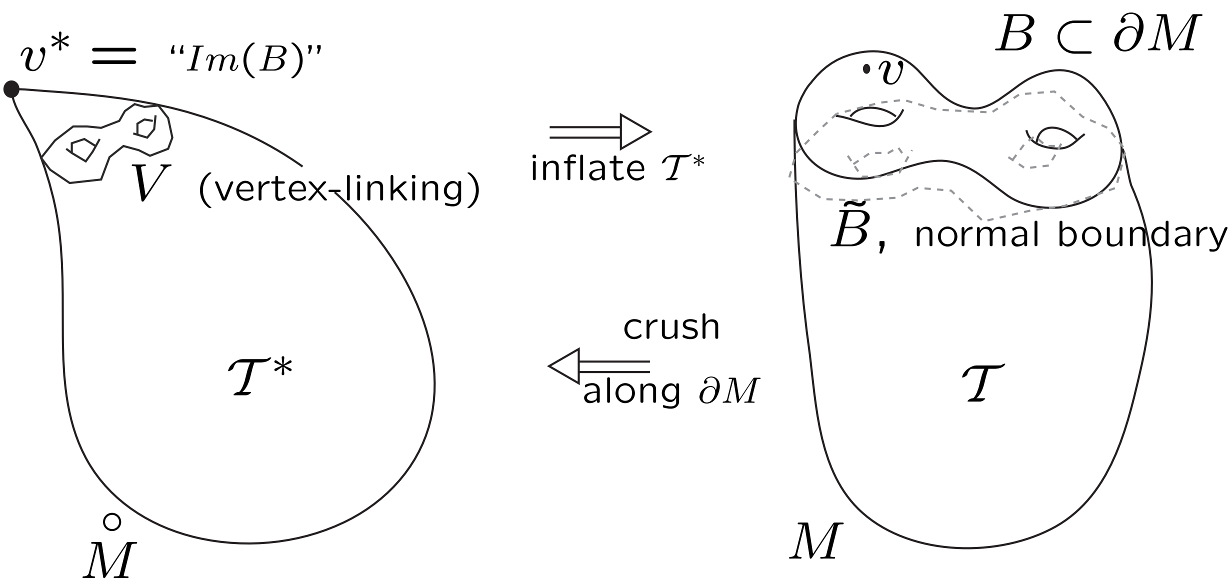
**Step 3.** Recapture $M$ with a new triangulation via an **inflation of an ideal triangulation of** $\mathring{M}$.

**Definition.** $\mathcal{T}^*$ an ideal triangulation of $\mathring{M}$. An **inflation of the ideal triangulation** $\mathcal{T}^*$ is one of a finite family of constructible *minimal-vertex* triangulations of $M$ having *normal boundary* so that crushing $\mathcal{T}$ along $\partial M$ gives $\mathcal{T}^*$.

• **minimal-vertex triangulation** of $M$ has all vertices in $\partial M$ and just one vertex in each component of $\partial M$.

• **normal boundary triangulation** of $M$ is a triangulation for which the frontier of a small regular neighborhood of each boundary component is normally isotopic to a normal surface.

**Theorem** (J-R). *For any compact* 3*–manifold* $M$ *with boundary, no component of which is a* 2*–sphere, and any ideal triangulation* $\mathcal{T}^*$ *of* $\mathring{M}$*, there exists an inflation* $\mathcal{T}$ *of* $\mathcal{T}^*$*.*



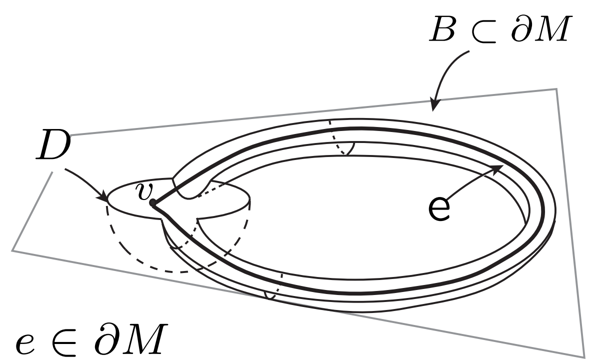

**Step 4.** Closed normal surfaces in ideal triangulations ⇔ closed normal surfaces in inflations.

**Theorem** (J-R). *$M$ a compact 3–manifold, $\mathcal{T}^*$ an ideal triangulation of $\overset{\circ}{M}$ and $\mathcal{T}$ an inflation of $\mathcal{T}^*$, then there is a bijective correspondence between the closed normal surfaces in $\mathcal{T}$ and the closed normal surfaces in $\mathcal{T}^*$.*

In particular,

- Boundary-linking surfaces in $\mathcal{T}$ $\Leftrightarrow$ vertex-linking surfaces in $\mathcal{T}^*$.

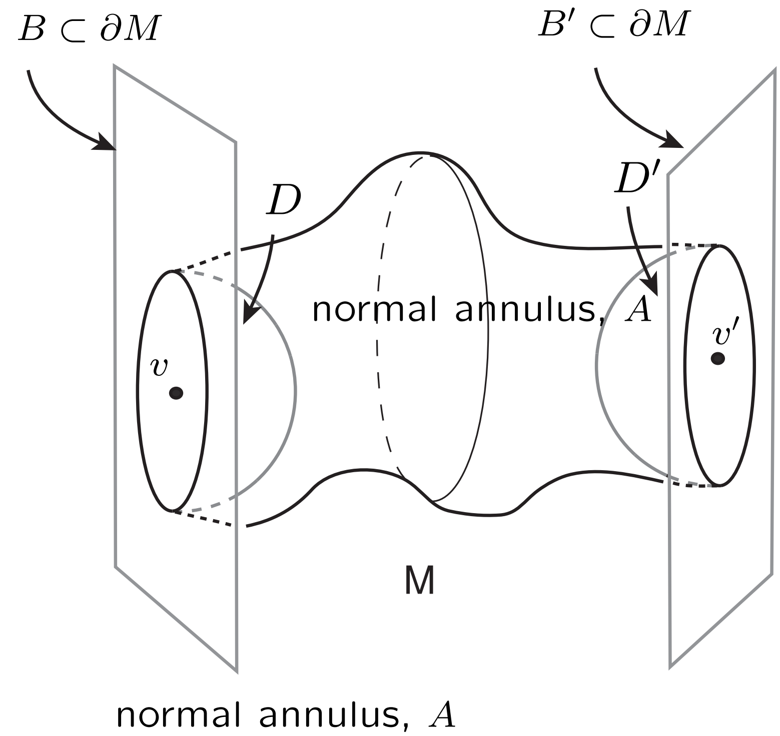
- Normal surface isotopic and not normally isotopic to boundary linking surface in $\mathcal{T}$ $\Leftrightarrow$ normal surface isotopic and not normally isotopic to vertex-linking surface in $\mathcal{T}^*$.




**CONCLUSION:** We have constructed a triangulation of $M$ that is 0–efficient and the only normal annuli with essential boundaries are *thin edge-linking*.

This completes the proof.□

**Remark.** It is possible that there are normal annuli with trivial boundary which are not thin edge-linking.




**Curious Fact.** $M$ a compact 3–manifold, $\mathcal{T}$ a 0–efficient triangulation of $M$. If $e$ is an edge of $\mathcal{T}$ with vertices $v \in B$ and $v' \in B'$, then a small regular neighborhood of $e$ is <u>normally isotopic</u> to a thin edge-linking annulus.