EFFICIENT TRIANGULATIONS

OF

THREE-MANIFOLDS

with

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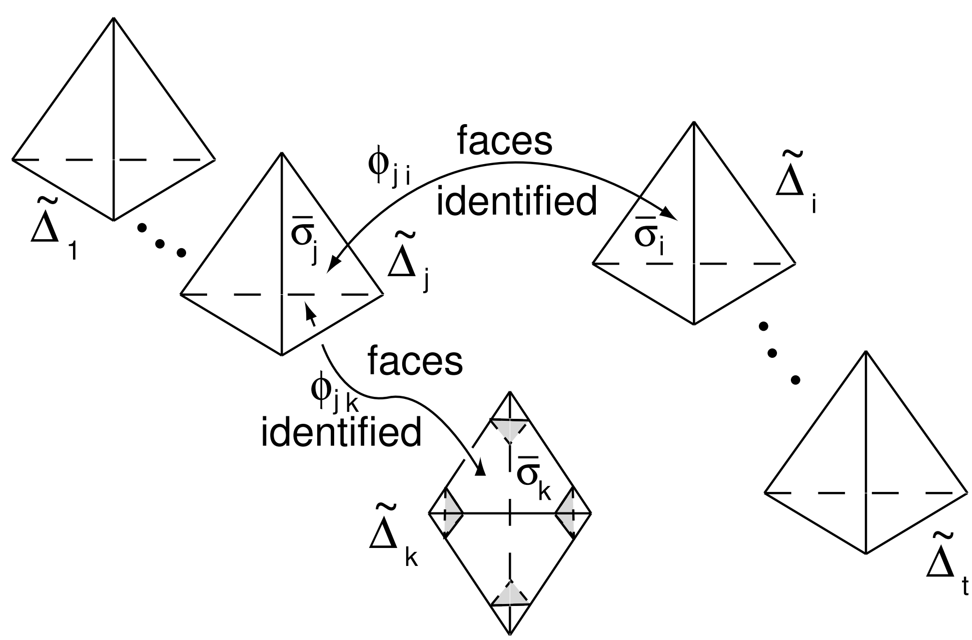
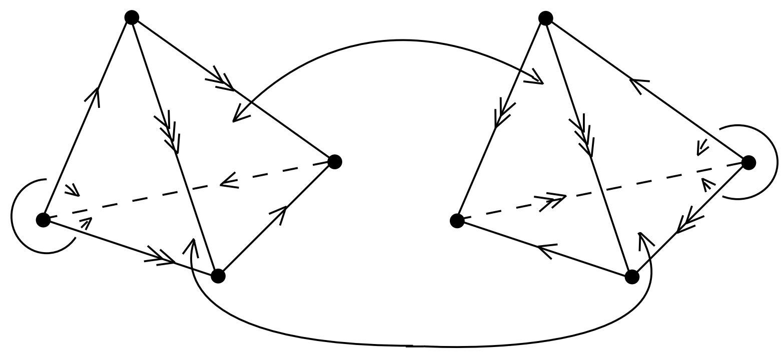
TRIANGULATIONS

$\mathbf{\Delta} = \{\tilde{\Delta}_1, \ldots, \tilde{\Delta}_t\}$ is a pairwise-disjoint collection of oriented, tetrahedra.

Φ is a family of affine isomorphisms pairing faces of the tetrahedra in $\mathbf{\Delta}$

$\phi \in \Phi$, then ϕ is an orientation-reversing affine isomorphism from a face $\sigma_i \in \tilde{\Delta}_i$ to a face $\sigma_j \in \tilde{\Delta}_j$, possibly $i = j$.





L(3,1)

$\boldsymbol{\Delta}/\Phi$ is a 3–manifold (except possibly at the images of the vertices of the $\tilde{\boldsymbol{\Delta}}_i$).

We collect all this information into a single symbol $\mathcal{T}$ and set $|\mathcal{T}| = \boldsymbol{\Delta}/\Phi$.

Link of each vertex a 2–sphere or a 2–cell, then $M = |\mathcal{T}|$ oriented 3–manifold, possibly with boundary, and call $\mathcal{T}$ a **triangulation** of M.

All triangles form **vertex-linking surface**, which in this case each component is either a 2–sphere or a disk.

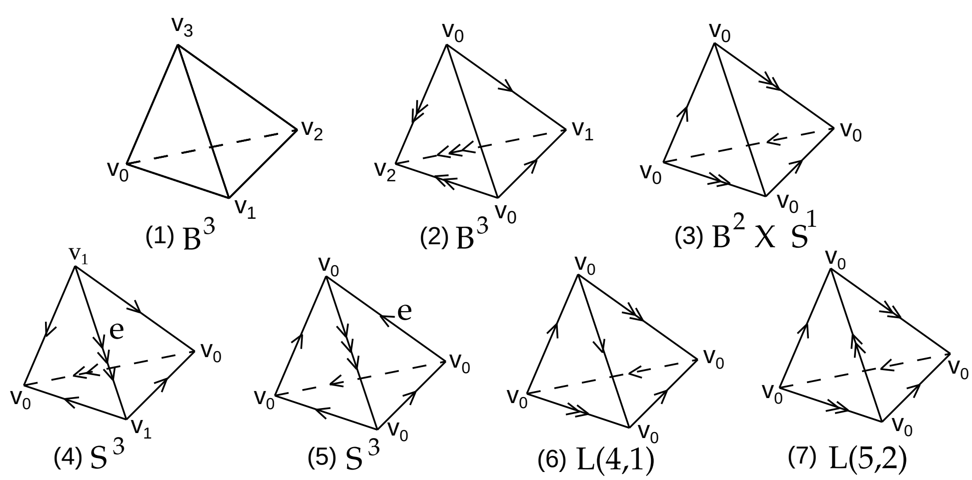


FIGURE: Five (5) orientable 3–manifolds and seven (7) distinct triangulations from a single tetrahedron.

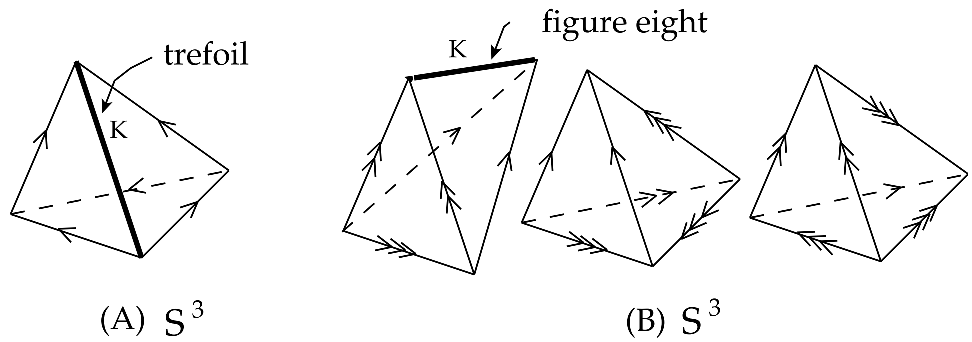


FIGURE: One-vertex triangulations of S^3 having the "trefoil knot" (A) as an edge and the "figure eight knot" (B) as an edge.

Theorem. *Given any knot in S^3, there is a one-vertex triangulation of S^3 having the knot as an edge of valence one.*

- Census of distinct triangulations up to six tetrahedra

- (Matveev et al) Census of all *minimal* triangulations up to nine tetrahedra.

Link of some vertex is a closed surface, distinct from the 2–sphere, we say $\mathcal{T}$ is an **ideal triangulation** of the 3–manifold $M = |\mathcal{T}| \setminus |\mathcal{T}^{(0)}|$.

All triangles form **vertex–linking surface**, which in this case may have genus ≥ 1, the **index** of the **ideal vertex**.

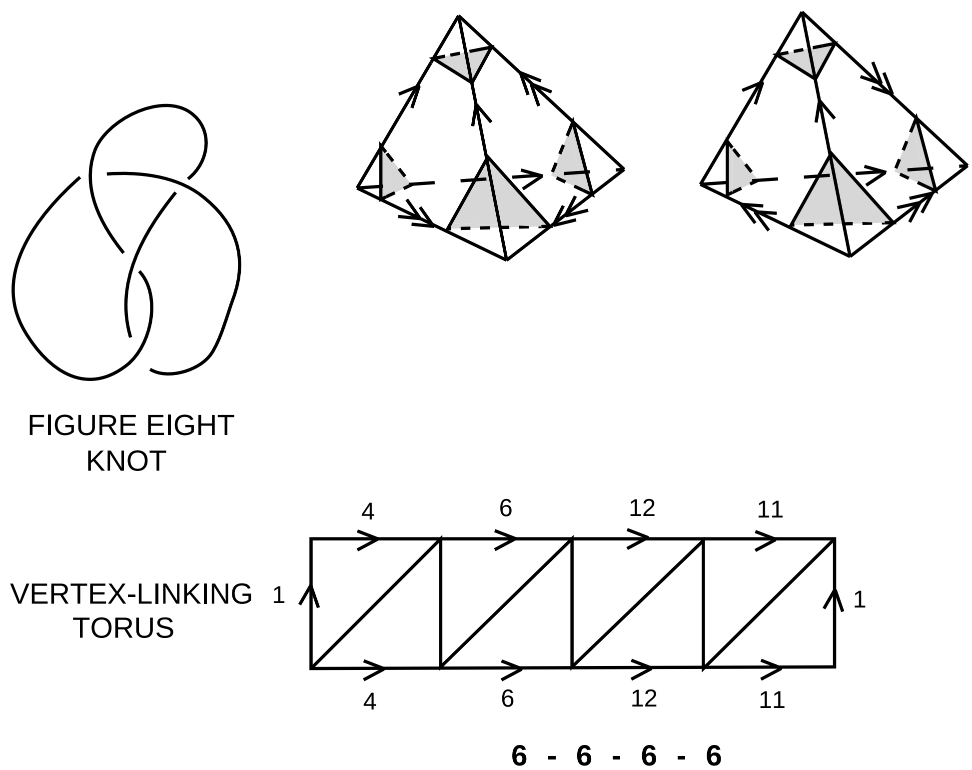


FIGURE: Ideal triangulation of Figure Eight knot complement in S^3

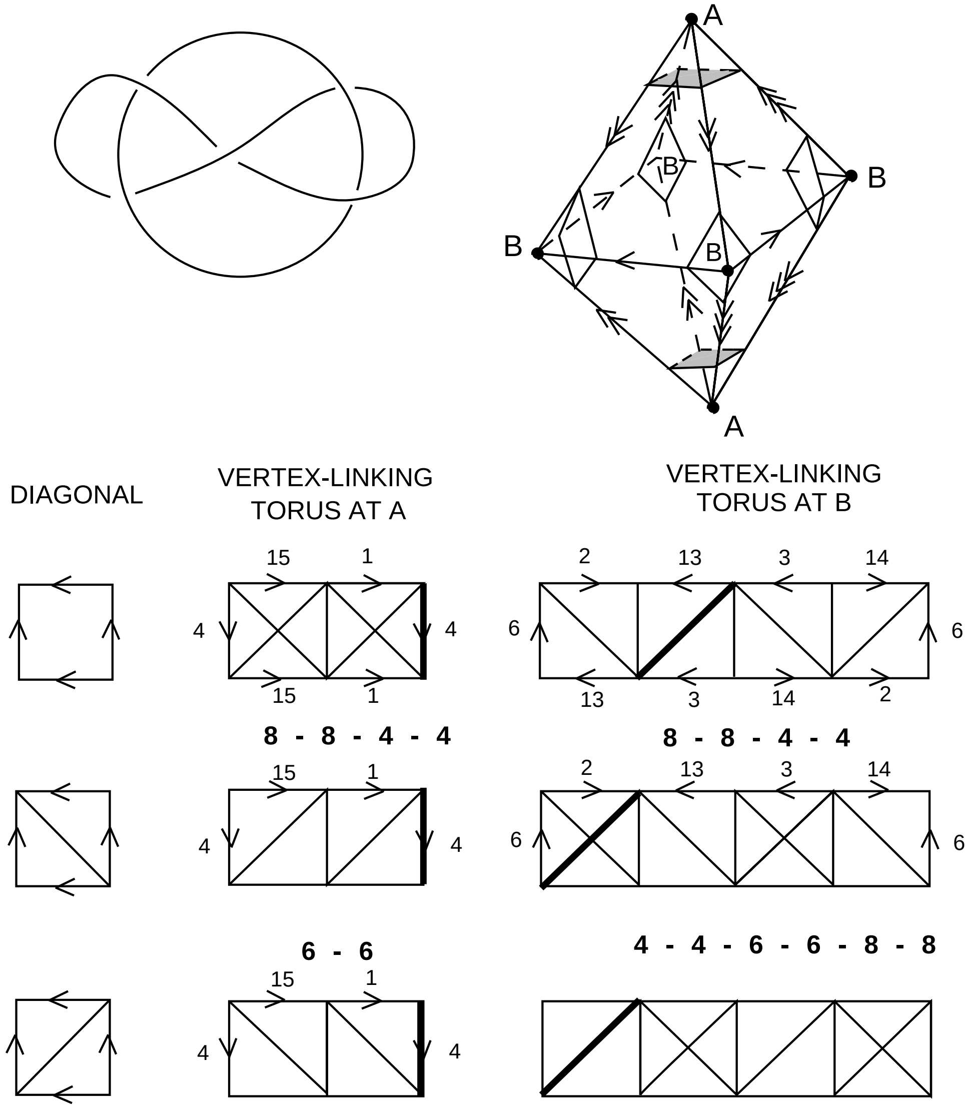


FIGURE: Ideal triangulation of Whitehead link complement in S^3.

QUESTION: “How common are ideal triangulations?”

Theorem. *Given any knot (link) in S^3, there is an ideal triangulation of its complement (with meridional slope an edge slope having length* ≤ 2*).*

Theorem. *Suppose M is a compact, irreducible, ∂–irreducible, 3–manifold. Then $\overset{\circ}{M}$ admits an ideal triangulation.*

Remark. If M is given with a triangulation $\mathcal{T}$ having t tetrahedra, then an ideal triangulation $\mathcal{T}^*$ of $\overset{\circ}{M}$ can be constructed having $< t$ tetrahedra.

LAYERED TRIANGULATIONS

Layered solid torus

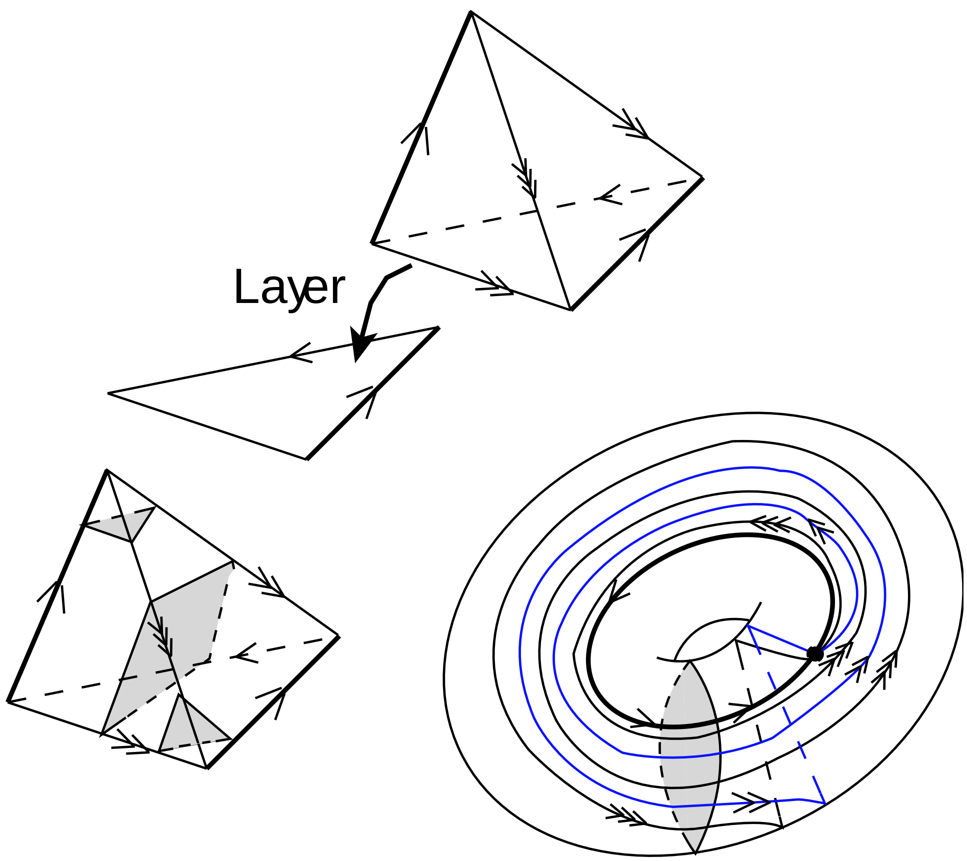


FIGURE: One-tetrahedron Triangulation of solid torus.

Layered solid torus (con't)

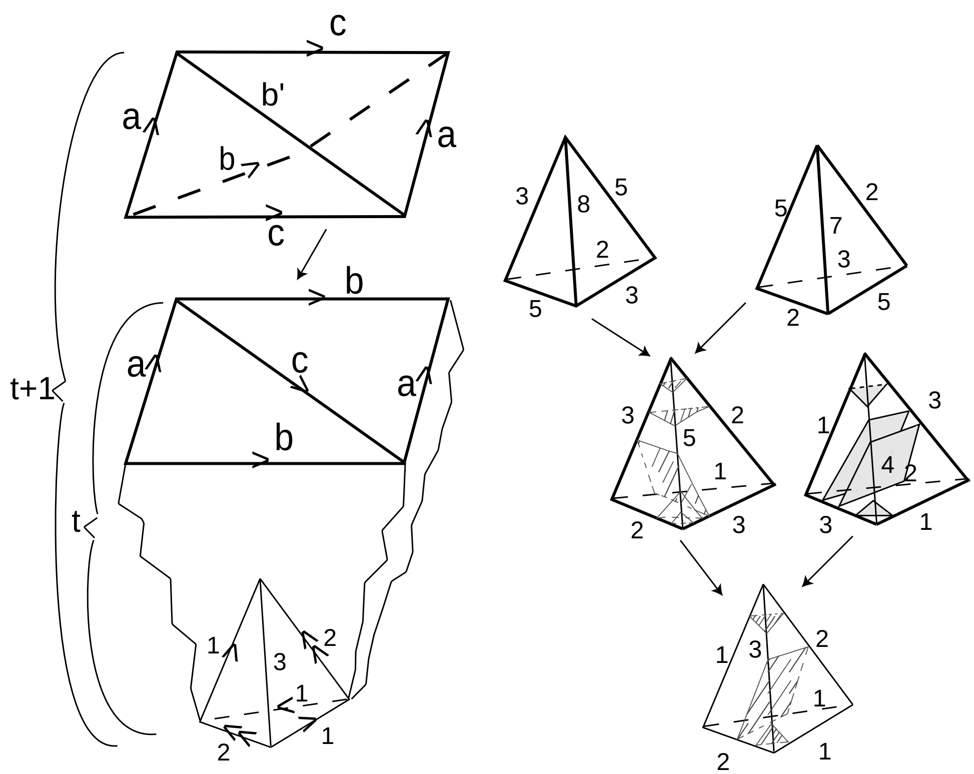
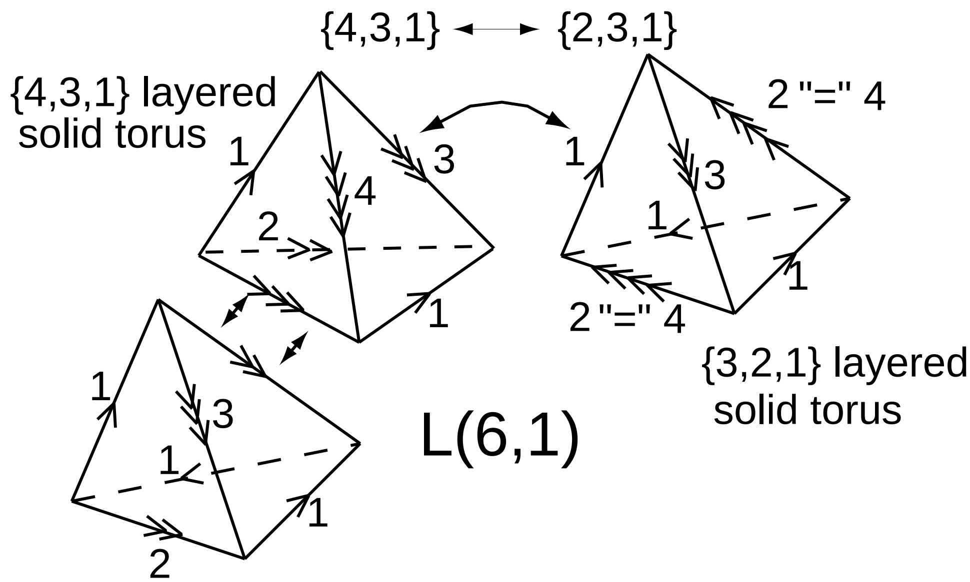


FIGURE 5. Layered solid torus.

FIGURE: Layered triangulation of a solid torus.

Theorem. *Given any one-vertex triangulation* $\mathcal{T}_\partial$ *on the boundary of a solid torus, there is a unique minimal layered triangulation* $\mathcal{T}$ *of the solid torus extending* $\mathcal{T}_\partial$*.*

Layered Triangulations of Lens Spaces



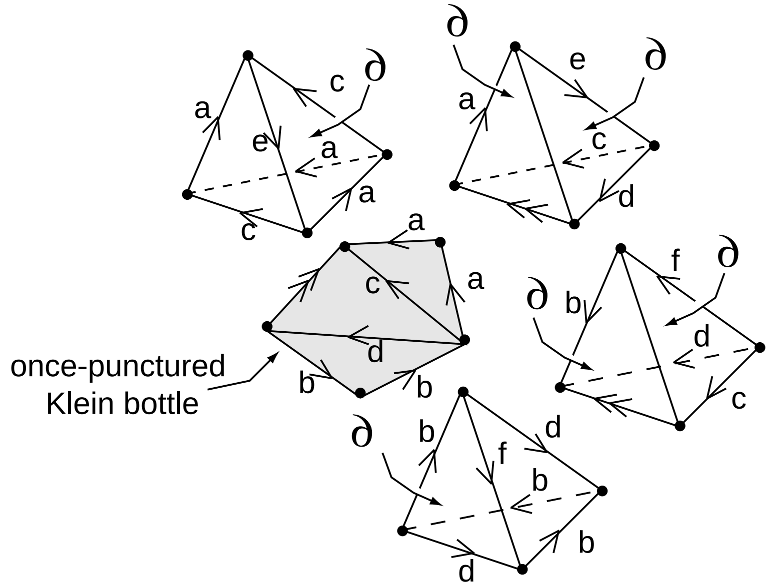
EXAMPLE: Layered triangulation of the Lens Space $L(6, 1)$.

Theorem. *Every lens space admits a layered triangulation.*

Type I. $\{3, 2, 1\} \leftrightarrow \{q, p + q, p\} = L(3p + q, p)$

Type II. $\{3, 2, 1\} \leftrightarrow \{q, p, p+q\} = L(3p+2q, p+q)$

Layered Triangulations of Handlebodies



genus 2 handlebody - layered triangulation

EXAMPLE: Minimal triangulation of genus 2 handlebody (t the number of tetrahedra, $t = 3g-2$ for minimal triangulation of genus g handlebody.)

Theorem. *Given any one-vertex triangulation $\mathcal{T}_\partial$ on the boundary of a handlebody, there is a layered triangulation $\mathcal{T}$ of the handlebody extending the triangulation $\mathcal{T}_\partial$.*

QUESTION: "How common are one-vertex triangulations?"

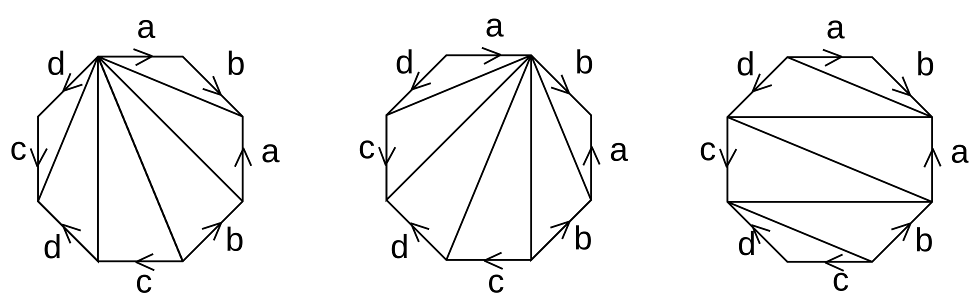


FIGURE: One-vertex (minimal) triangulations of genus 2, closed, orientable surface.

Theorem. *Any closed orientable* 3*–manifold admits a one-vertex (layered) triangulation.*

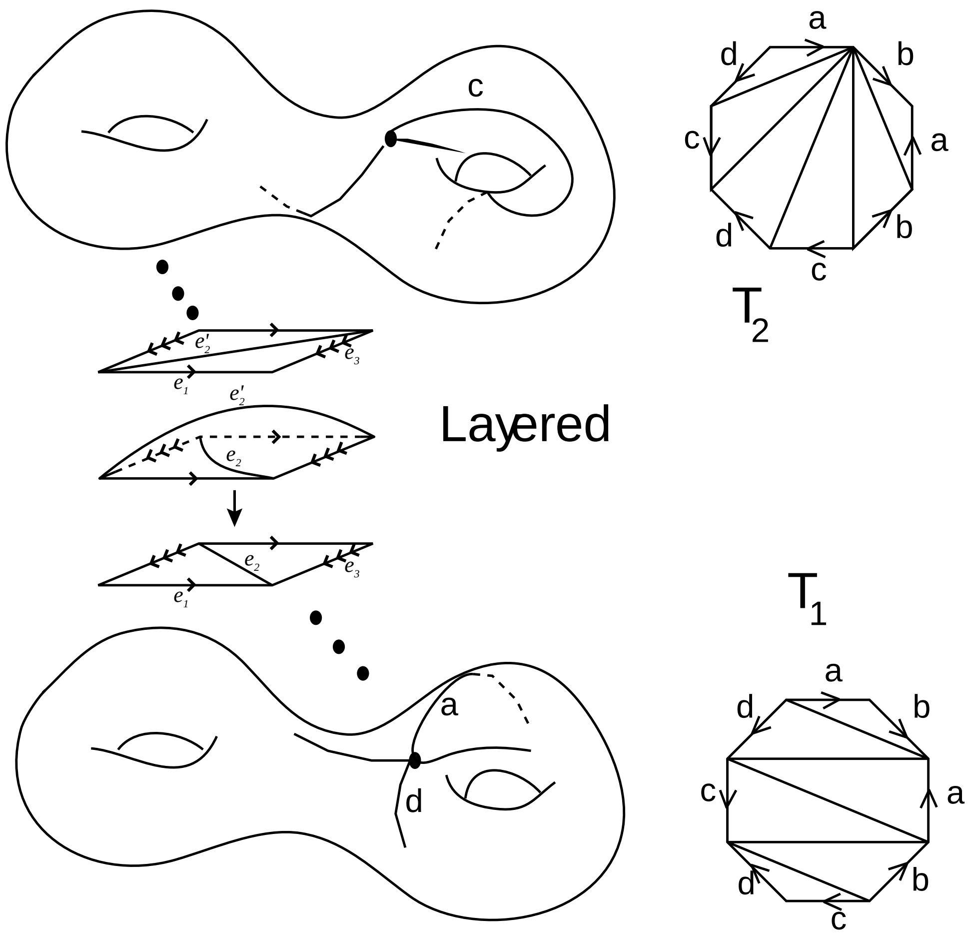


FIGURE: One-vertex triangulations of closed, orientable 3–manifolds from layered triangulations of handlebodies.

NORMAL SURFACE THEORY

A triangulation $\mathcal{T}$ of a 3–manifold M determines a class of surfaces in M called **normal** or **almost normal surfaces**.

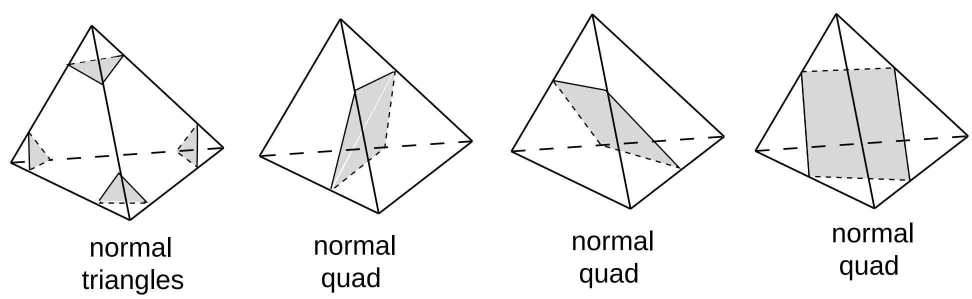


FIGURE: Elementary normal disks in Normal Surface Theory.

EXAMPLES:

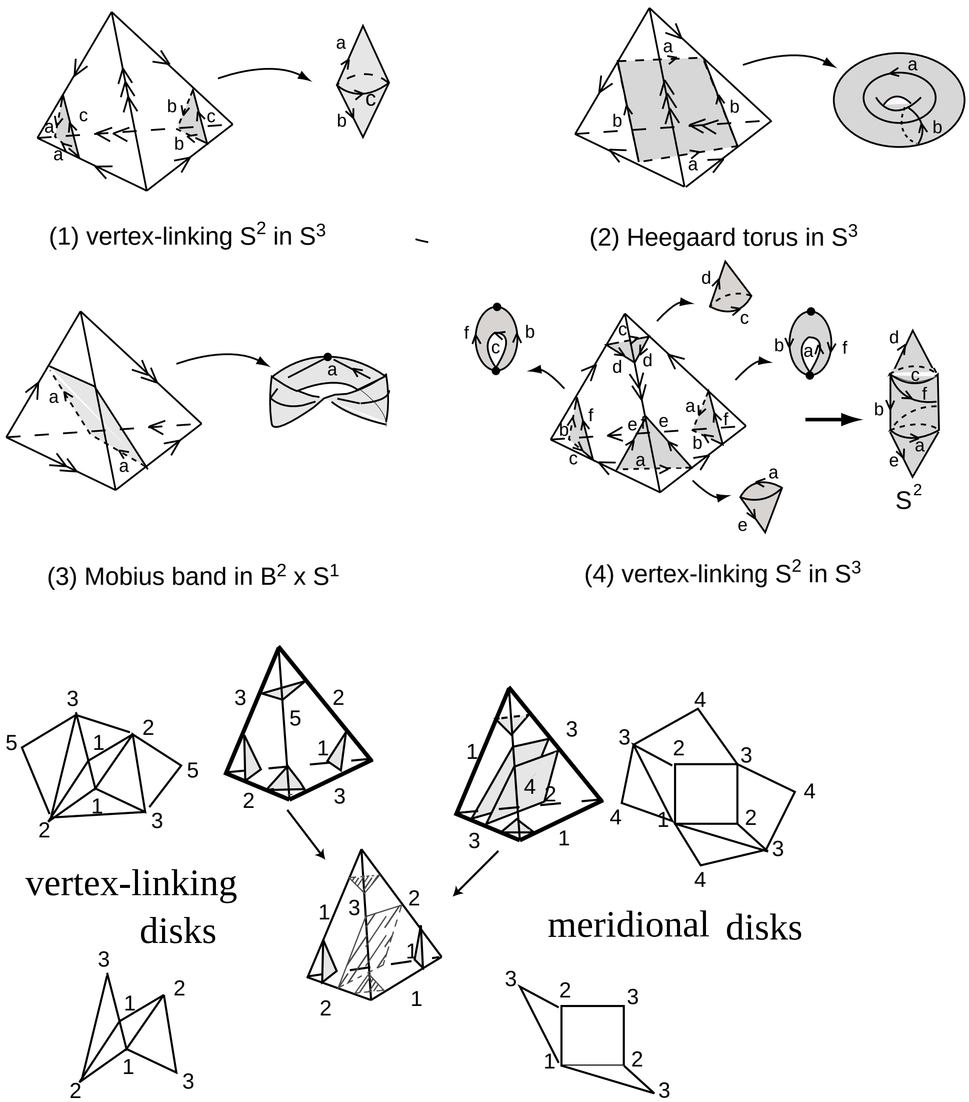


FIGURE: Some Examples of normal surfaces.

Existence of Normal Surfaces

- *vertex-linking* - collection of all elementary triangles; normally isotopic to the boundary of a small regular neighborhood of the vertices.

- *edge-linking* - basically, normally isotopic to a small regular neighborhood of an edge. This is sometimes a torus (always for one-vertex triangulations) and sometimes is not normal.

- **Theorem.** M a 3–manifold and F an essential embedded surface in M. Then for any triangulation of M there is an essential, embedded normal surface in M (which is homeomorphic to F).

Existence of Normal Surfaces (con't)

- **Theorem.** M a 3–manifold, $\mathcal{T}$ a triangulation of M and K and L are disjoint subcomplexes. Then there is a normal surface separating K from L. These are called **splitting surfaces**.

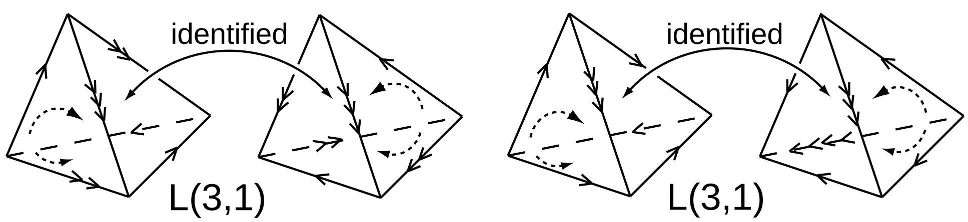
  Remark. If $M = H \cup_S H'$ is a Heegaard splitting of the 3–manifold M and $\mathcal{T}$ is the "natural" two-vertex triangulation of M coming from this Heegaard splitting of M, then the Heegaard surface S is a splitting surface with K and L being the two vertices of $\mathcal{T}$.

- $\mathcal{T}$ a triangulation of the irreducible 3–manifold M, then either $M = S^3$ or there is a normal 2–sphere bounding a 3–cell in M and containing all the vertices of $\mathcal{T}$.

0–EFFICIENT TRIANGULATIONS

Closed 3–manifolds

The triangulation $\mathcal{T}$ of the closed, orientable 3–manifold M is **0–efficient** iff the only normal, embedded 2–spheres are vertex-linking.



A. Two solid tori; 0-efficient B. Solid torus and 3-cell; not 0-efficient

Proposition. *Suppose M is a closed, orientable 3–manifold. If M has a 0–efficient triangulation, then M is irreducible and $M \neq \mathbb{R}P^3$. Furthermore, either the triangulation has one vertex or M is S^3 and the triangulation has precisely two vertices.*

Remark. Suppose the closed, orientable 3–manifold M has a 0–efficient triangulation $\mathcal{T}$.

1. If $\mathcal{T}$ has an edge that bounds an embedded disk, then $M = S^3$. ($M \neq S^3$, then every edge is knotted.)

2. If $\mathcal{T}$ has an edge of order one, then $M = S^3$.

3. If $\mathcal{T}$ has a face which is a cone, then $M = S^3$.

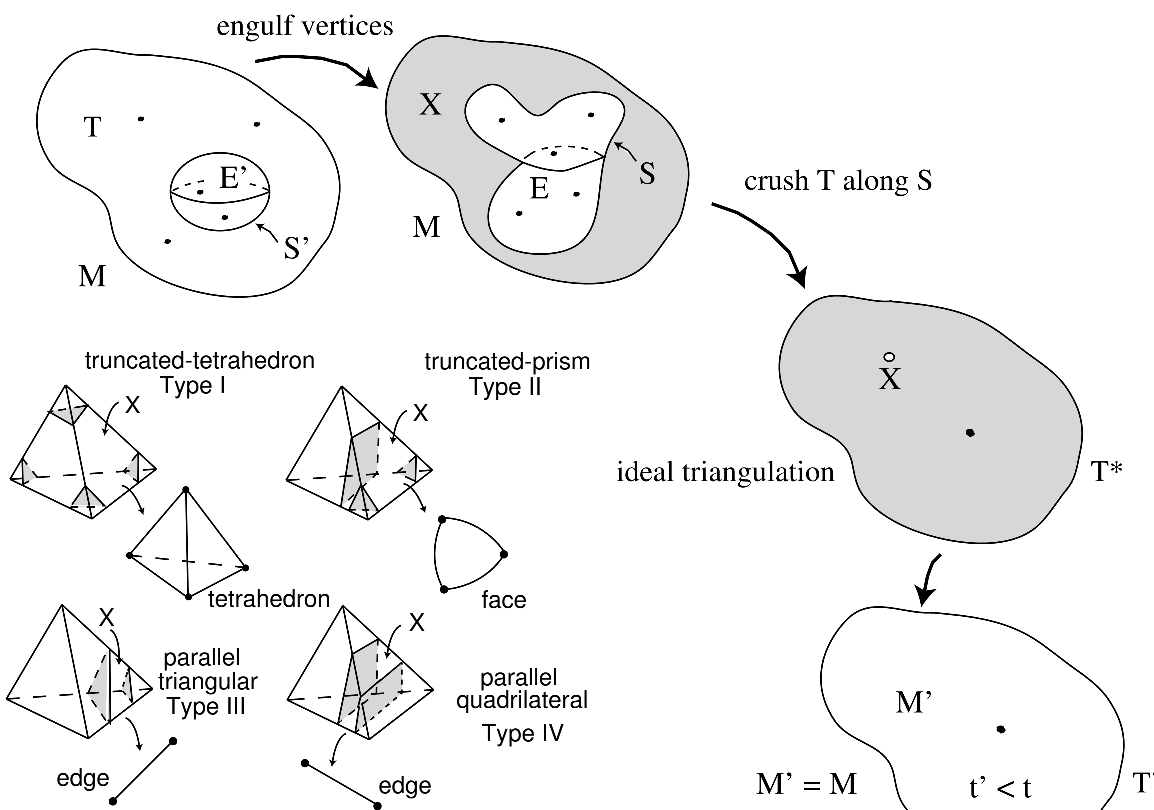
4. If $\mathcal{T}$ has an edge of order two, then $\mathcal{T}$ can be reduced or $M = L(3,1)$ or $L(4,1)$, and

5. If $\mathcal{T}$ has an edge of order three, the $\mathcal{T}$ can be reduced or $M = L(5,2)$ or $\mathcal{T}$ contains, as a subcomplex, the two-tetrahedron, geodesic layered triangulation $\{4,3,1\}$ of the solid torus.

Question: “How common are 0–efficient triangulations?”

Theorem. *A closed, orientable, irreducible* 3–*manifold distinct from* $\mathbb{R}P^3$ *has a* 0–*efficient triangulation.*

Outline of Proof.



Theorem. *Given a closed, orientable* 3*–manifold* M*, there is an algorithm to construct a finite family of* 3*–manifolds,* $M_1, \ldots, M_n$*, so that* $M = M_1 \# \ldots \# M_n$*, where* $M_i, i = 1, \ldots, n$*, either has a* 0*–efficient triangulation or can be shown to be homeomorphic with one of* $S^3, S^2 \times S^1, \mathbb{R}\mathrm{P}^2$*, or the lens space,* $L(3, 1)$*.*

Theorem (3**–Sphere Recognition Problem).** *Given a* 3*–manifold* M*, it can be decided if* M *is homeomorphic with the* 3*–sphere.*

Bounded 3–manifolds

A triangulation of a compact 3–manifold with non-empty boundary is **0–efficient** if and only if all normal disks are vertex-linking.

Proposition. *Suppose M is a compact, orientable 3–manifold with nonempty boundary, no component of which is a 2–sphere. If M has a 0–efficient triangulation, then there are no normal 2–spheres and M is irreducible and ∂–irreducible. Furthermore, the triangulation has all its vertices in ∂M and has precisely one vertex in each boundary component.*

Theorem. *A compact, orientable, irreducible, ∂–irreducible 3–manifold with nonempty boundary has a 0–efficient triangulation.*

Ideal Triangulations

The ideal triangulation $\mathcal{T}$ of $\mathring{M}$ is a 0**–efficient ideal triangulation** if and only if there are no normal 2–spheres.

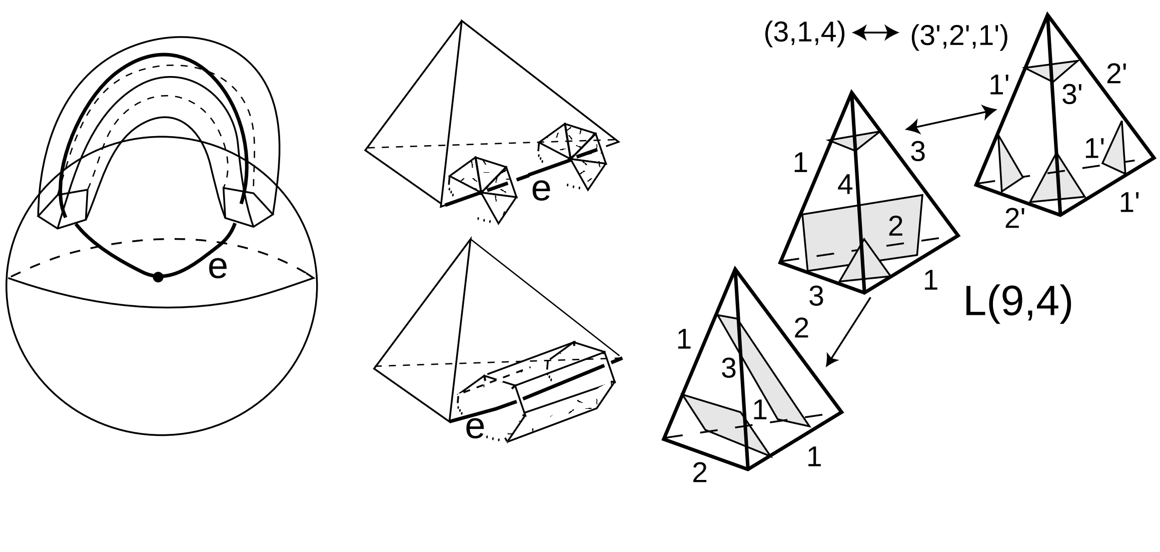
Theorem. *Suppose M is a compact, irreducible, ∂–irreducible 3–manifold. Then any ideal triangulation of $\mathring{M}$ can be modified to a 0–efficient ideal triangulation of $\mathring{M}$.*

1–EFFICIENT TRIANGULATIONS

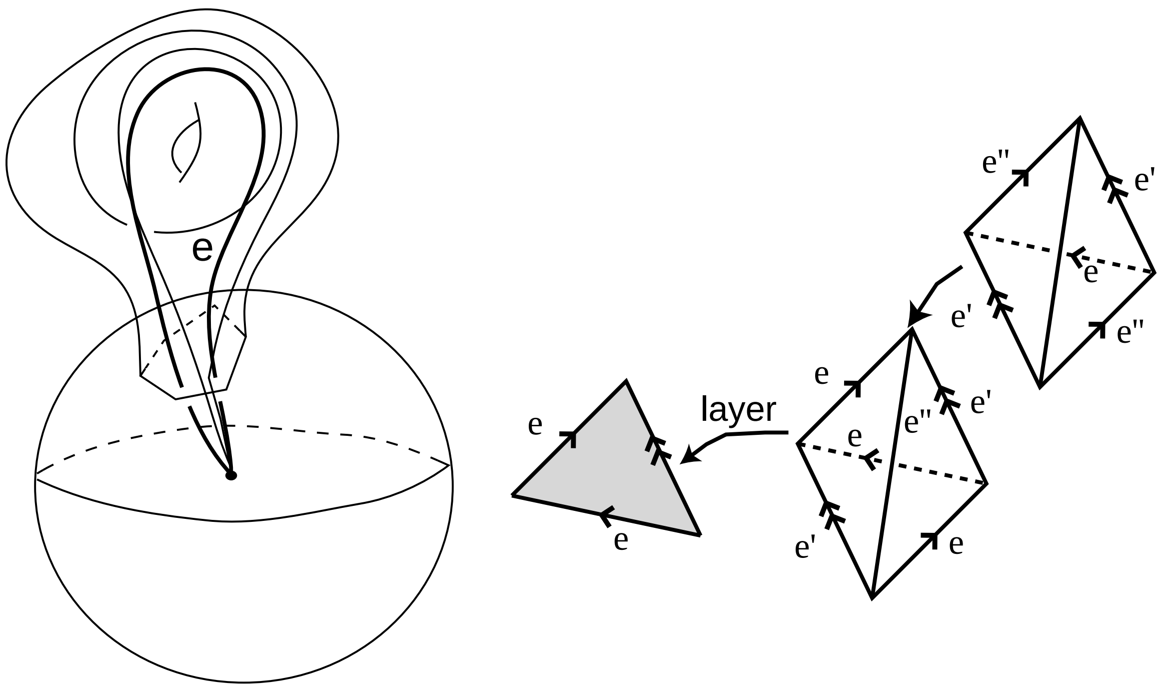
<u>Closed 3–manifolds</u>

M a closed, orientable 3–manifold, $\mathcal{T}$ a triangulation of M. A <u>normal</u> torus, embedded in M is:

- a **thin edge-linking torus** if it is normally isotopic to a regular neighborhood of an edge in $\mathcal{T}$.



- a **thick edge-linking torus** if it is normally isotopic to the boundary of a geodesic layered solid torus, which is a subcomplex of $\mathcal{T}$.



A <u>normal</u> torus is **edge-linking** iff it is either a thin edge-linking torus or a thick edge-linking torus.

A triangulation $\mathcal{T}$ of a closed, orientable 3–manifold is **1–efficient** iff it is 0–efficient and every normal torus is edge-linking.

Proposition. *Suppose M is a closed, orientable 3–manifold. If M has a 1–efficient triangulation, then M is irreducible, atoridal and $M \neq \mathbb{R}P^3$.*

Remarks:

1. A minimal layered triangulation of a lens space (distinct from $\mathbb{R}P^3$) is 1–efficient.

2. There are infinitely many 0–efficient triangulations of a lens space (distinct from $\mathbb{R}P^3$); however, there are only finitely many 1-efficient triangulations.

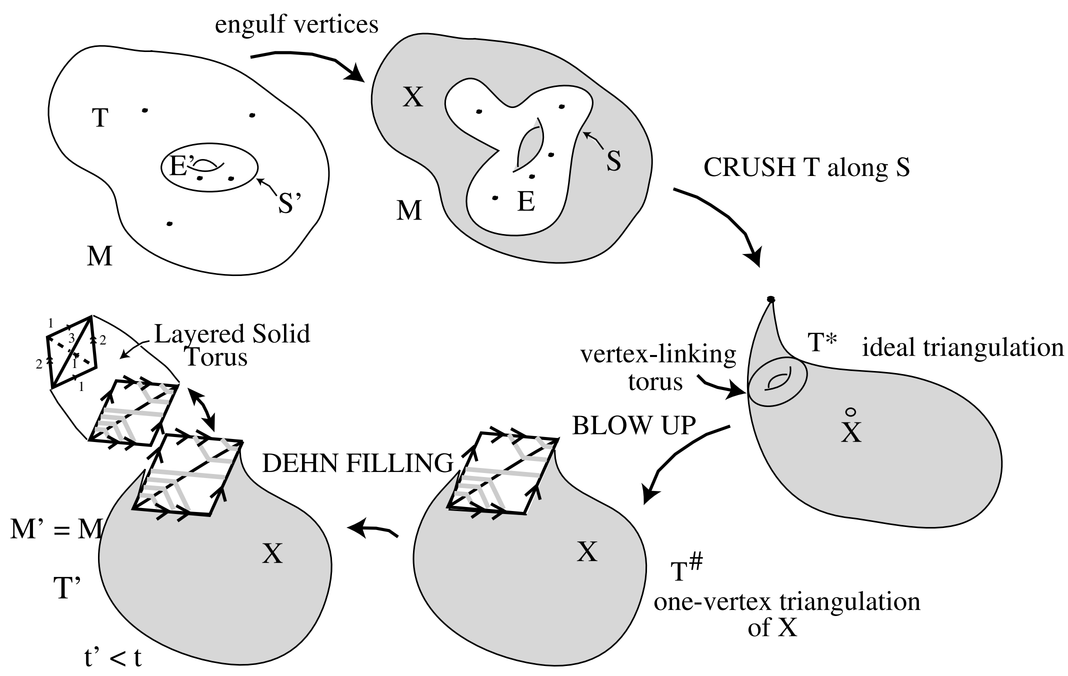
   A triangulation $\mathcal{T}$ of the 3–manifold M is said to be **minimal** if for any triangulation $\mathcal{T}'$ of M, we have $t \leq t'$, where t and t' are the number of tetrahedra of $\mathcal{T}$ and $\mathcal{T}'$, respectively.

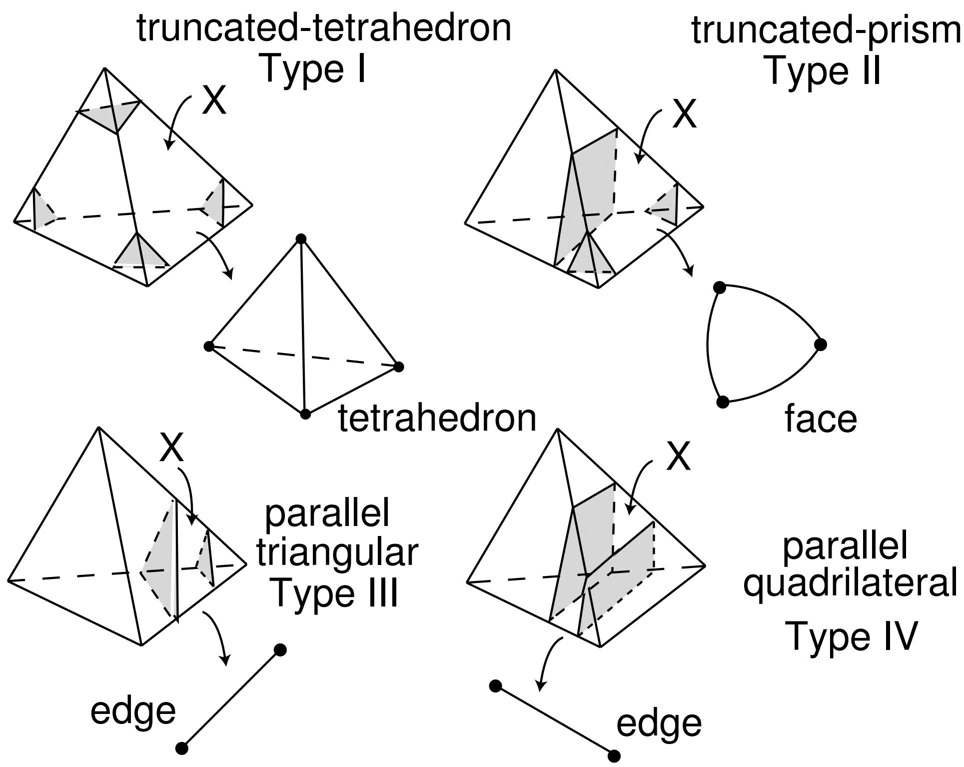
3. A minimal triangulation of an irreducible 3–manifold (distinct from $\mathbb{R}P^3$) must be 0–efficient. We do not know if a minimal triangulation of an irreducible, atoridal 3–manifold (distinct from $\mathbb{R}P^3$) is 1–efficient.

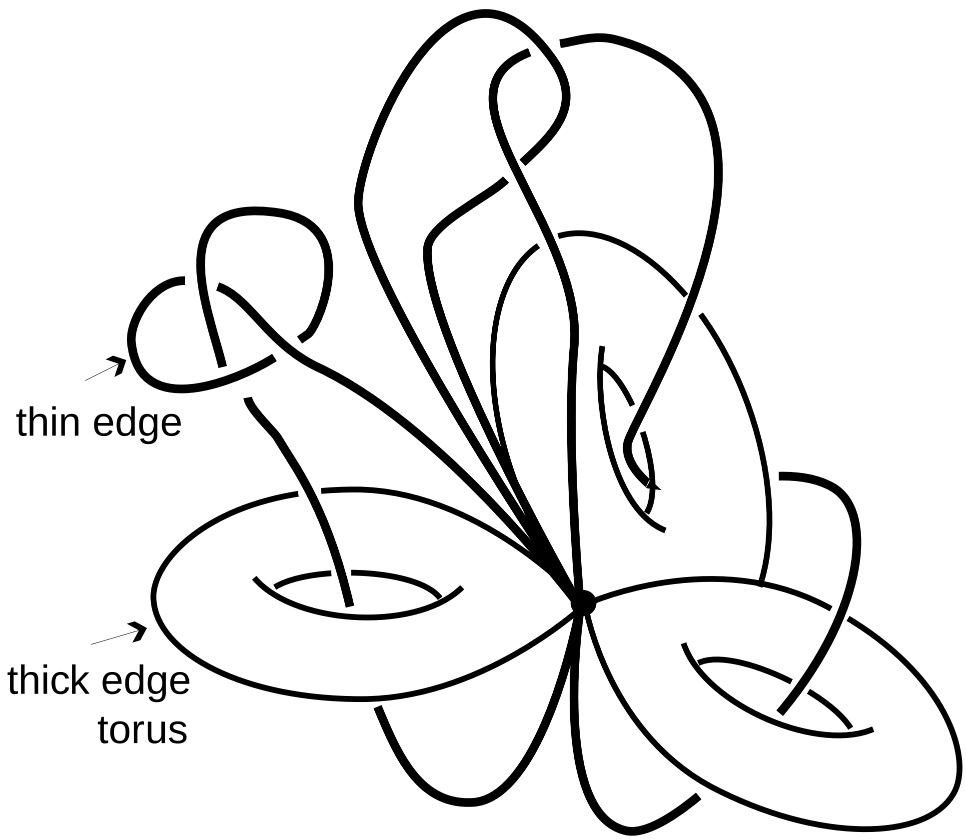
Question: "How common are 1–efficient triangulations?"

Theorem. *Suppose M is a closed, orientable, irreducible, atoridal 3–manifold. Then any triangulation of M can be modified to a 1–efficient triangulation or it can be shown that M is homeomorphic to S^3, a lens space, or a small Seifert fiber space.*

Outline of Proof.




truncated-tetrahedron
Type I
X
tetrahedron
truncated-prism
Type II
X
face
X
parallel
triangular
Type III
edge
X
parallel
quadrilateral
Type IV
edge



(a) Thick and thin edges

Ideal Triangulations

The ideal triangulation $\mathcal{T}$ of $\overset{\circ}{M}$ is a 1–**efficient ideal triangulation** if and only if it is 0–efficient and the only normal tori are vertex-linking.

Theorem. *Suppose* M *is a compact, irreducible,* ∂*–irreducible, atoridal* 3*–manifold* M *with boundary a collection of tori. Then any ideal triangulation of* $\overset{\circ}{M}$ *can be modified to a* 1*–efficient ideal triangulation of* $\overset{\circ}{M}$*.*