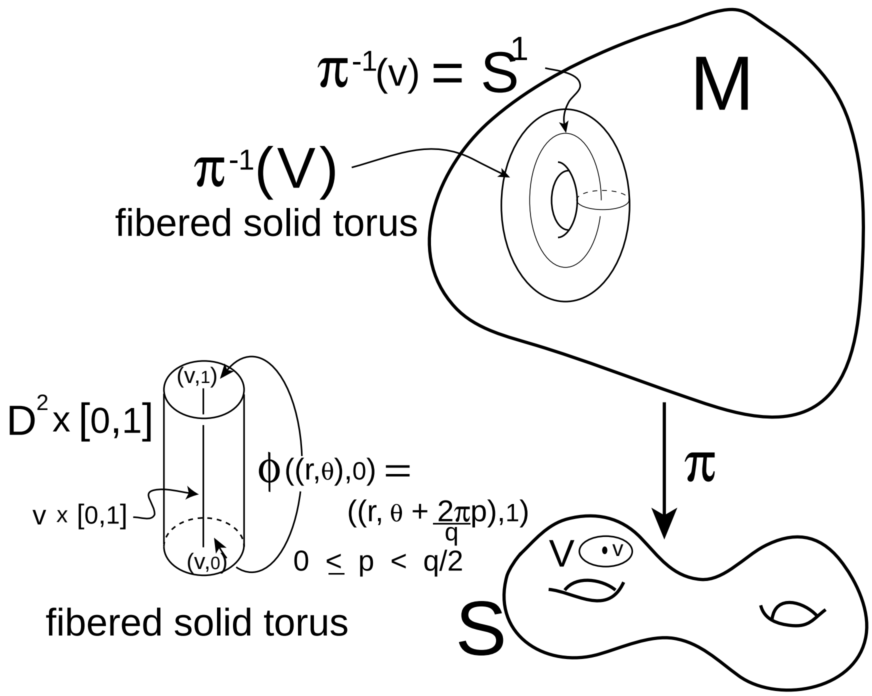
VA. The JSJ Decomposition of a 3–manifold

Abstract: This lecture will give a brief introduction to Seifert fibered 3–manifolds and the existence and uniqueness theorem of Jaco-Shalen and Johannson for the JSJ Decomposition of a 3–manifold.

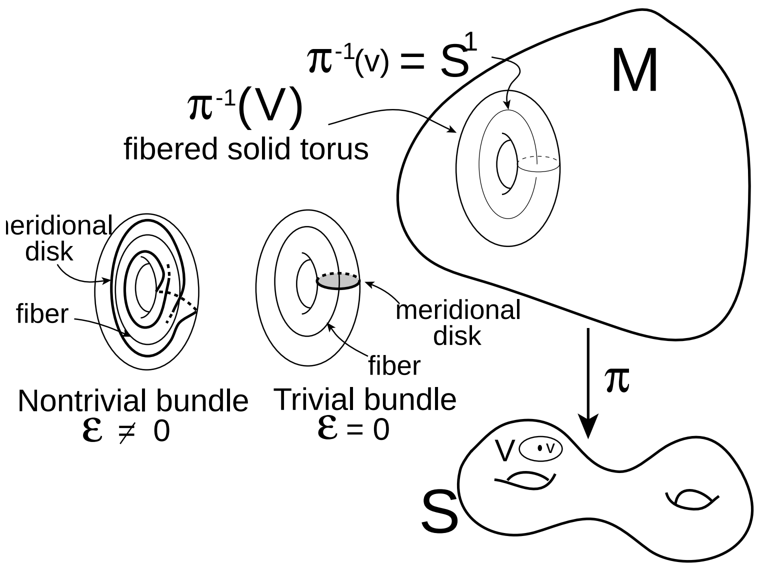
Definition. essential surface, atoridal 3–manifold, fibered solid torus, Seifert fibered 3–manifold, regular fiber, exceptional fiber, multiplicity



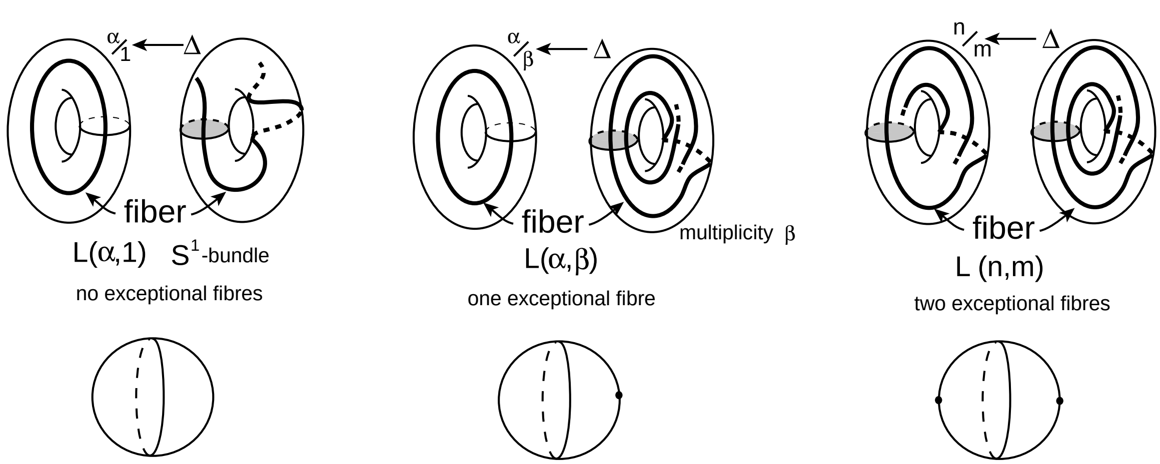
• A compact Seifert fibered 3–manifold has only finitely many exceptional fibers, all of which are in the interior of the 3–manifold.

EXAMPLES:

1. Circle bundles (S^1–bundles) over a surface



2. Lens spaces



Seifert Invariants:

$\partial M \neq \emptyset; \quad M(\pm g, b : 0, \alpha_1/\beta_1, \ldots, \alpha_K/\beta_K)$

$\partial M = \emptyset; \quad M(\pm g, 0 : \varepsilon, \alpha_1/\beta_1, \ldots, \alpha_K/\beta_K)$

$g =$ genus of base; "$+$" is orientable base, "$-$" is nonorientable base; b is number of boundary components; ε is Euler number of the bundle; α_i/β_i denotes the i^{th} Dehn filling.

NOTE:

1. $\partial M \neq \emptyset$, then $b \neq 0$ and $\varepsilon = 0$

2. $\partial M = 0$, then $b = 0$ and $\varepsilon \in \mathbb{Z}$

3. the i^{th} exceptional fiber has multiplicity β_i.

Incompressible Surfaces in Seifert-fibered Spaces:

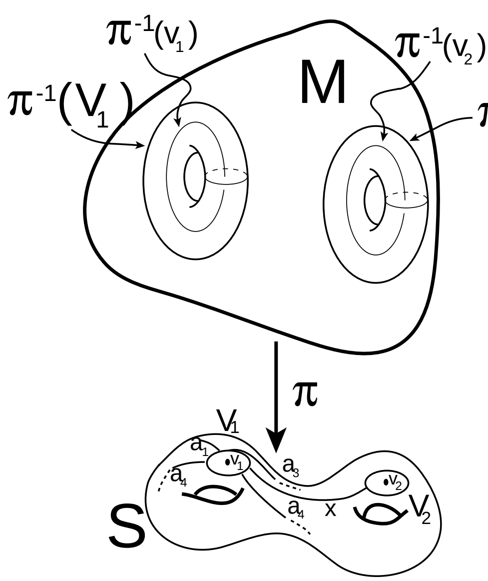
Definitions: vertical surfaces (Examples: boundary tori, vertical annuli, tori, Möbius bands and Klein bottles); bundle over S^1, semi-bundle over S^1; horizontal surfaces (Examples: fibers in bundles and semi-bundles), surfaces parallel into boundary

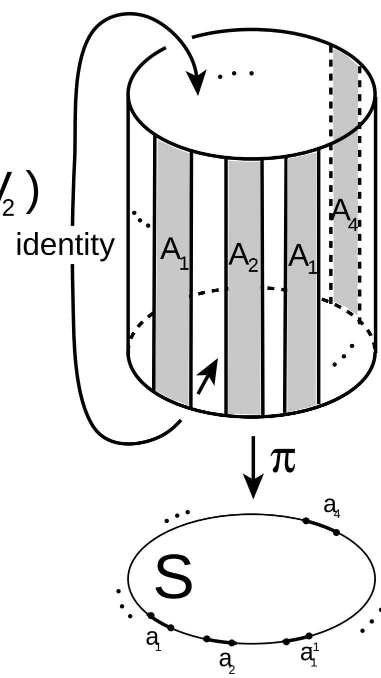
Lemma. *An incompressible surface in a $3-$manifold with torus boundary is $\partial-$incompressible or a boundary parallel annulus.*

Lemma. *An incompressible surface in a solid torus is either a meridional disk or a boundary parallel annulus.*

Theorem. *A connected, incompressible surface in a Seifert-fibered space is either vertical or horizontal.*

Proof.





Corollary. *A horizontal two-sided surface in a Seifert-fibered space, M, is a fiber in a fibration of M as a bundle or semi-bundle over S^1.*

Corollary. *A Seifert fibered space is irreducible except $S^2 \times S^1$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$.*

Corollary. *An incompressible annulus in a Seifert fiber space is isotopic to a vertical annulus after possibly changing the fibering in $S^1 \times S^1 \times I$ or the twisted $I-$bundle over the Klein bottle.*

Fundamental Group:

"+ base"

$$\pi_1(M) \equiv \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_e, d_1, \ldots, d_b, h : a_i h a_i^{-1} h^{-1},\ b_i h b_i^{-1} h^{-1},\ c_j h c_j^{-1} h^{-1},\ d_k h d_k^{-1} h^{-1}, c_j^{\alpha_j} h^{\beta_j},\ h^{\varepsilon} \prod [a_i, b_i] \cdot c_1 \cdots c_e \cdot d_1 \cdots d_b \rangle.$$

"- base"

$$\pi_1(M) \equiv \langle a_1, \ldots, a_g, , c_1, \ldots, c_e, d_1, \ldots, d_b, h : a_i h a_i^{-1} h,\ c_j h c_j^{-1} h^{-1},\ d_k h d_k^{-1} h^{-1},\ c_j^{\alpha_j} h^{\beta_j}, h^{\varepsilon} a_1^2 \cdots a_g^2 \cdot c_1 \cdots c_e \cdot d_1 \cdots d_b \rangle.$$

Theorem (Seifert,(1933); Waldhausen,(1967)). *Two Seifert-fibered spaces are homeomorphic iff they have the same Seifert invariants with the exceptions:*

(a) *Solid Torus;* $M(0,1:\alpha/\beta) \approx M(0,1:\alpha'/\beta')$

(b) *Twisted I–bundle over the Klein Bottle;* $M(0,1:1/2,1/2) \approx M(-1,1:)$

(c) *Lens Spaces (including S^3 and $S^2 \times S^1$);* $M(0,0:\alpha_1/\beta_1,\alpha_2/\beta_2)$

(d) *Prism manifolds;* $M(0,0;1/2,-1/2,\alpha/\beta) \approx$

$M(-1,0;\beta/\alpha)$

(e) *Double of Twisted I–bundle over Klein Bottle:* $M(0,0;1/2,1/2,-1/2,-1/2) \approx M(-2,0;)$.

Corollary. *Given two Seifert fibred spaces via their Seifert invariants, it can be decided if they are homeomorphic.*

THE JSJ DECOMPOSITION

Theorem (Jaco-Shalen,1977; Johannson,1977). *A closed, irreducible, orientable 3–manifold M has a unique (up to isotopy) submanifold Σ so that each component of Σ is Seifert fibered and each component of $M\setminus \mathring{\Sigma}$ is atoridal.*

REMARK: There is a (more complex) version for manifolds with nonempty boundary.