IV B. An Algorithm to Construct the Prime Decomposition of a 3–manifold

Abstract: This lecture will use the new methods of crushing triangulations to give an algorithm for constructing the prime decomposition of a 3–manifold.

Definition. 0–efficient triangulation

REMARKS:

1. If a 3–manifold has a 0-efficient triangulation, it is irreducible and not $\mathbb{R}P^3$.

2. A 0–efficient triangulation of a closed 3–manifold has one vertex or the manifold is S^3 and the triangulation has two vertices.

3. Any closed, irreducible 3–manifold (except $\mathbb{R}P^3$) has a 0–efficient triangulation; in fact, (except for $\mathbb{R}P^3$) a minimal triangulation of an irreducible 3–manifold is 0–efficient.

EXAMPLES: Infinite family of two-vertex 0–efficient triangulations of S^3; infinite family of 0–efficient layered triangulations of a lens space ($\neq \mathbb{R}P^3$).

Theorem (Jaco-Rubinstein,1998). *(Construction of Prime Decomposition) Given a* 3*–manifold* M *with triangulation* $\mathcal{T}$*, there is an algorithm that constructs a prime decomposition* $M = M_1 \# \ldots \# M_n$ *of* M *or concludes* $M = S^3$*.*

Proof:(Outline)

Step 1. Can decide if $\mathcal{T}$ is 0–efficient. If YES - then check if $M = S^3$; if NO, then get normal non vertex-linking 2–sphere.

Step 2. Construct a punctured 3–cell P so that ∂P consists of normal, non-vertex-linking 2–spheres and P contains the vertices of $\mathcal{T}$ or show $M = S^3$.

Step 3. Enlarge P, if necessary, to a punctured 3–sphere-with-handles (also called P) so that the closure of each component of $M \setminus P$ has connected boundary a normal, non vertex-linking 2–sphere or show $M = S^3$.

Proof:(Outline continued)

Step 4. If X is a component of $M \setminus P$, then $\hat{X}$ is a factor in a connected sum decomposition of M.

Step 5. With $M, \mathcal{T}$ and an X as above, we can crush the triangulation $\mathcal{T}$ along the normal 2–sphere ∂X or show $\hat{X}$ is one of S^3, $\mathbb{R}P^3$ or $L(3,1)$.

Step 6. We have $M = q_1(S^2 \times S^1) \# \hat{X}_1 \# \ldots \# \hat{X}_{r_1}$, where each $\hat{X}_i$ has been shown to be either $S^3, \mathbb{R}P^3$ or $L(3,1)$ or has a triangulation $\mathcal{T}_i$ with fewer tetrahedra than $\mathcal{T}$.

Step 7. Check each triangulation $\mathcal{T}_i$ to see if it is 0–efficient. If YES - then check if $\hat{X}_i = S^3$; if NO, then follow Steps 2-6 with $\hat{X}_i$ in place of M, $\mathcal{T}_i$ in place of $\mathcal{T}$. The process MUST stop.

REMARKS:

1. Each M_i is given by a triangulation $\mathcal{T}_i$.

2. The triangulation $\mathcal{T}_i$ of M_i is 0–efficient or $M_i = \mathbb{R}P^3$ or $S^2 \times S^1$.

3. (Rubinstein, 1993) It can be decided if a 3–manifold with a 0–efficient triangulation is S^3.

4. $\sum|\mathcal{T}_i| \leq |\mathcal{T}|$ with equality iff $\mathcal{T}$ is 0–efficient.

5. We actually have:

$$M = p(\mathbb{R}P^3)\#q(S^2 \times S^1)\#M_1\# \ldots \#M_r,$$

   where $M_i \neq S^3$ has a 0–efficient triangulation and $n = p + q + r$.

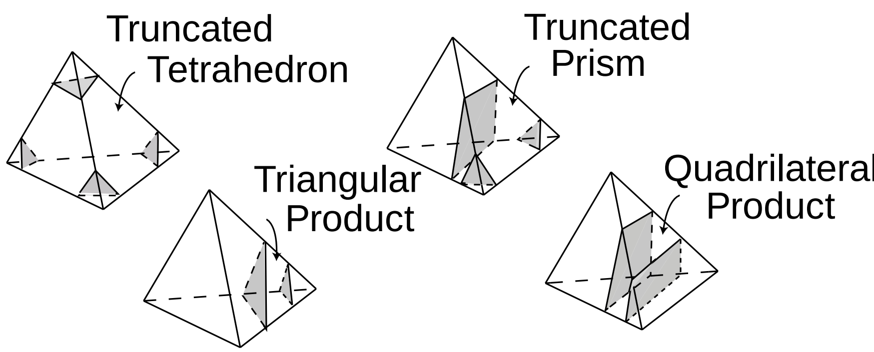
CRUSHING TRIANGULATIONS

Situation for **crushing a triangulation $\mathcal{T}$ of M along the normal surface S**

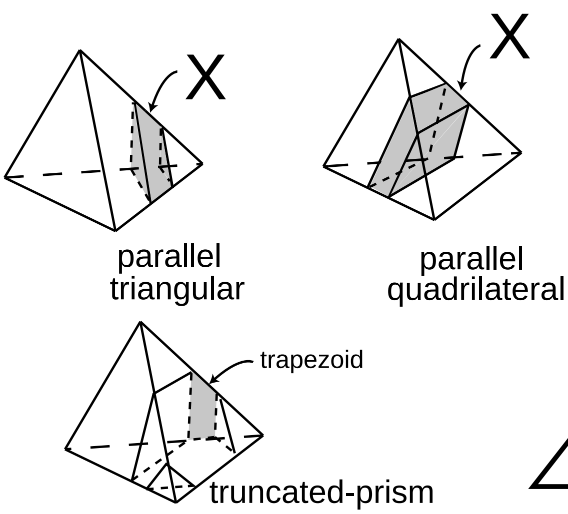
- M is a 3–manifold with triangulation $\mathcal{T}$
- S is a normal surface in M
- X is the closure of a component of the complement of S in M
- X contains no vertices of $\mathcal{T}$

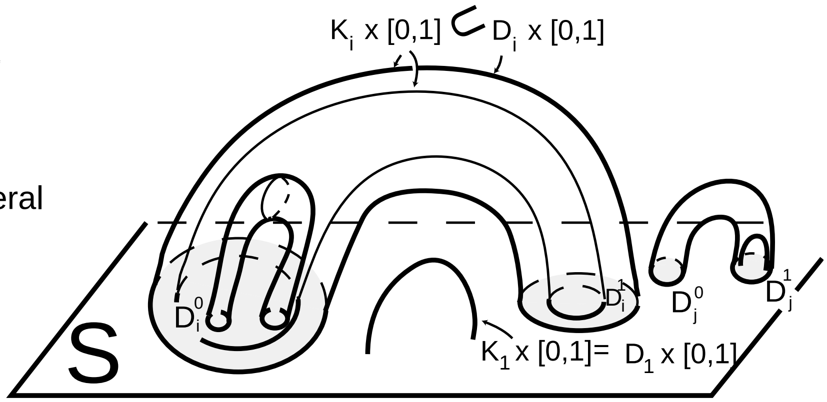
In this situation, we construct a particularly nice ideal triangulation of $\mathring{X}$.

• Induced Cell-decomposition on X

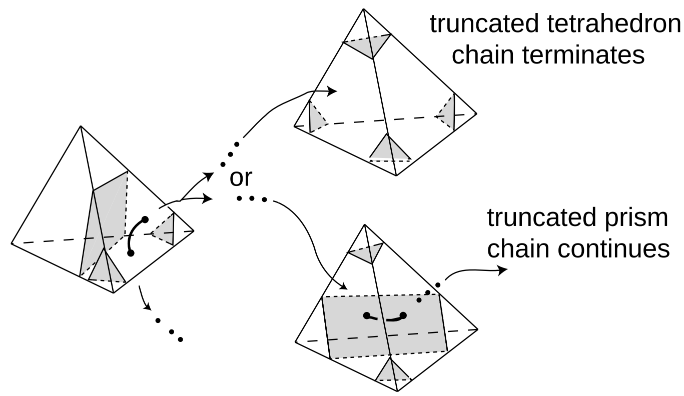


• Induced Product Region in X, $\mathcal{P}(\mathcal{C})$ and $\mathcal{P}(X)$

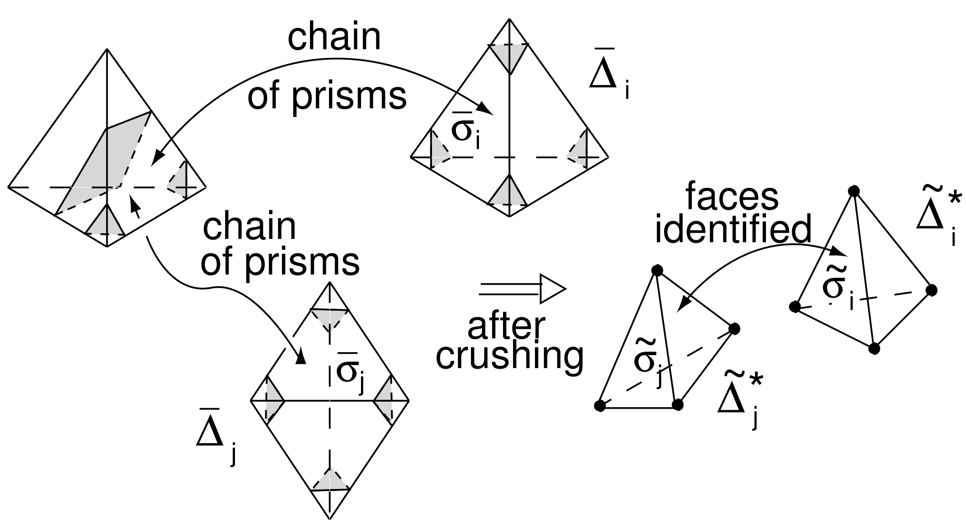




- Chains of truncated prisms in X

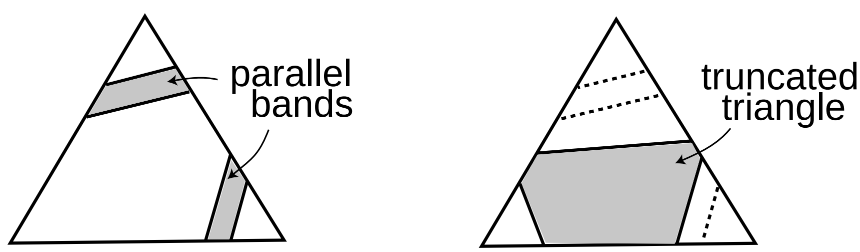


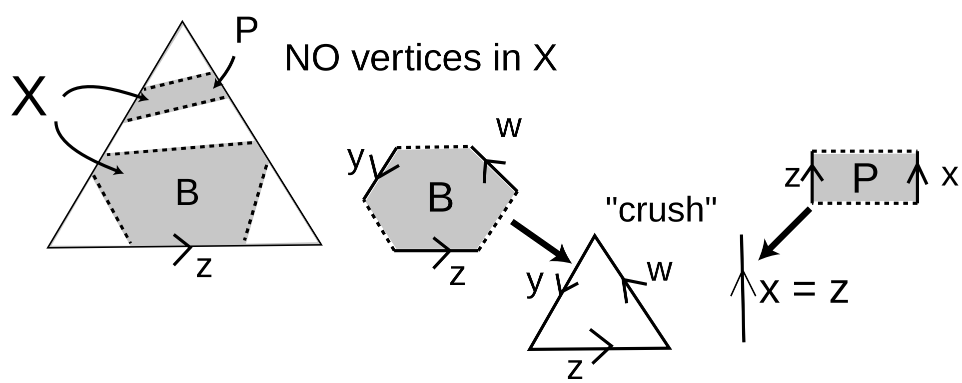
- (New) Face identifications



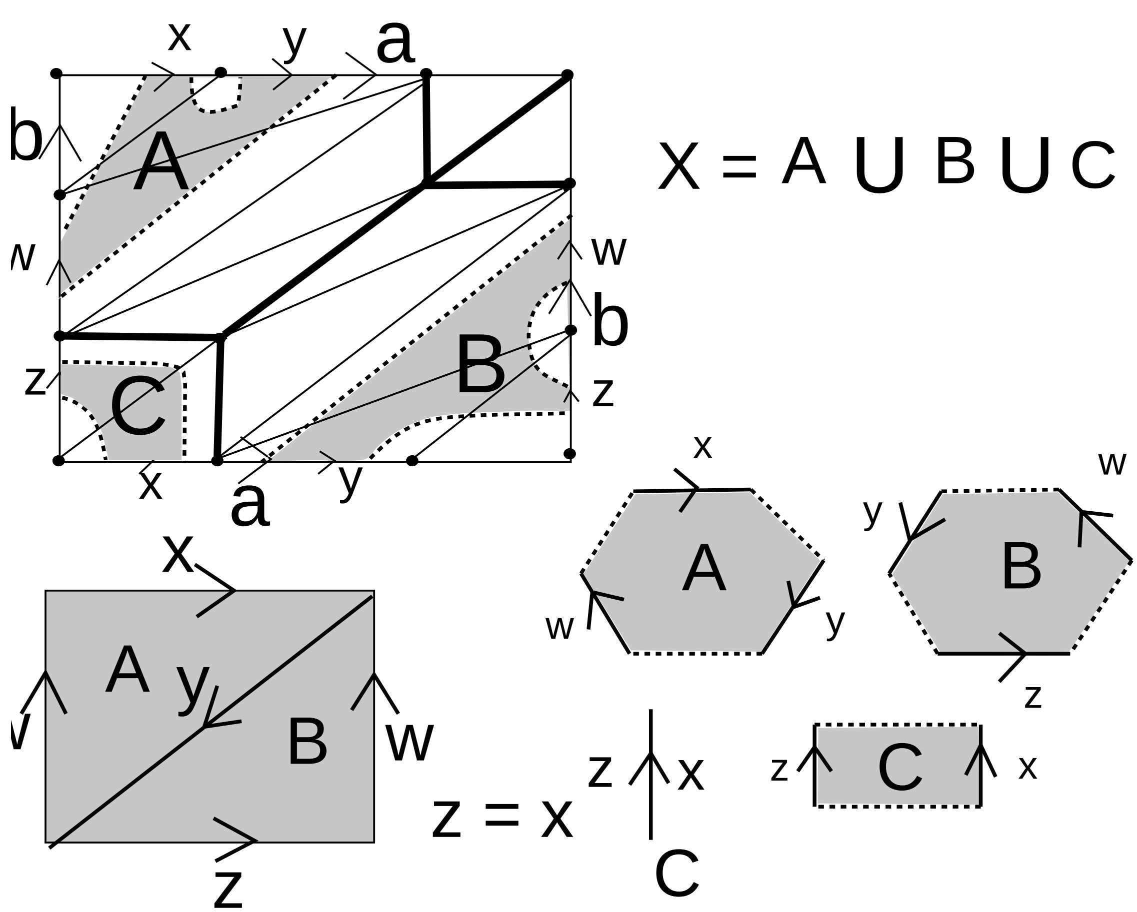
2-DIMENSIONAL ANALOGUE

• Induced Cell-decomposition on X in the case S is a normal curve in a triangulation of a 2–manifold





- Crush Triangulation of a torus



(Other) Constructions of Irreducible Decompositions

Theorem (Haken,1961). *Given a* 3*–manifold* M*, there is an algorithm to construct a pairwise disjoint collection of embedded* 2*–spheres in* M *giving an irreducible decomposition of* M*.*

Theorem (Jaco-Tollefson,1989). *For any triangulation* $\mathcal{T}$ *of the* 3*–manifold* M*, there is a simplex face of* $\mathcal{P}(M,\mathcal{T})$*, whose vertices are independent, normal* 2*–spheres embedded in* M *giving an irreducible decomposition of* M*.*

Theorem (Jaco-Rubinstein,1990). *Given a* 3*–manifold* M*, there is an algorithm to construct a maximal collection of pairwise, disjoint normal* 2*–spheres in* M*.*

Theorem (Jaco-Reeves, 1998). *For any triangulation* $\mathcal{T}$ *of the* 3*–manifold* M *any maximal collection of pairwise disjoint normal* 2*–spheres among the vertex solutions gives an irreducible decomposition of* M*.*