III B. Normal Surface Theory: Parameterizations and Algorithms

Abstract: This lecture explores some of the applications of normal surfaces to decision problems in 3–manifold topology.

NORMAL SOLUTION SPACE

Definition. normal coordinates, matching equations, $\mathcal{S}(M, \mathcal{T})$, quadrilateral conditions.

EXAMPLES:

1. one-tetrahedron torus;

2. two-tetrahedron $S^2 \times S^1$;

3. three-tetrahedron torus with two-vertices of Burton

QUADRILATERAL SOLUTION SPACE

Definition. quad coordinates, edge-equations, $\mathcal{S}^Q(M, \mathcal{T})$

EXAMPLES:

1. one-tetrahedron torus;
2. two-tetrahedron $S^2 \times S^1$;
3. three-tetrahedron torus with two-vertices of Burton

DISCUSS: Projection map from Normal Solution Space to Quad Solution Space (Example: layered torus or lens space).

Lemma. *If M is a compact 3–manifold and $\mathcal{T}$ is a triangulation of M having t tetrahedra, e edges and v vertices, then the dimension of $\mathcal{S}(M, \mathcal{T}) = t + e$ and dimension of $\mathcal{S}^Q(M, \mathcal{T}) = t + e - v$.*

Definition. normal equations, projective solution spaces ($\mathcal{P}(M, \mathcal{T})$ and $\mathcal{P}^Q(M, \mathcal{T})$), vertex solutions

Proposition. *Suppose $\mathcal{T}$ is a triangulation of the compact 3–manifold M. There is a unique (minimal) set of nonnegative integer solutions $[F_1], \ldots, [F_K]$ of the matching equations (for $\mathcal{T}$) so that if $[F]$ is any nonnegative integer solution of the matching equations, then*

$$[F] = \sum_{i=1}^{K} n_i [F_i],$$

where n_i is a nonnegative integer for all $i = 1, \ldots, K$.

NOTE: The same statement holds for solutions to the quad equations (for $\mathcal{T}$).

Definition. fundamental solutions, carrier, projective equivalence

Discuss finding vertex and fundamental solutions - complexity??

Lemma. *We have:*

- *A solution* $[F]$ *is a fundamental solution iff whenever* $[F] = [X] + [Y]$*, then* $[X] = 0$ *or* $[Y] = 0$.

- *A solution* $[F]$ *is a vertex solution iff whenever there are nonnegative integers* k, n *and* m *with* $k[F] = n[X] + m[Y]$*, then* $[X]$ *or* $[Y]$ *is projectively equivalent to* $[F]$.

NOTE: The vertex solutions are fundamental solutions.

Definition. regular exchange, quadrilateral conditions (compatible normal surfaces), geometric addition, boundary length

Proposition. *M a 3–manifold, F and G compatible normal surfaces in M, then*

1. $[F + G] = [F] + [G]$,

2. $wt(F + G) = wt(F) + wt(G)$,

3. $\chi(F + G) = \chi(F) + \chi(G)$, *and*

4. $L(\partial(F + G)) = L(\partial F) + L(\partial G)$.

Definition. fundamental and vertex surfaces, projection to Quad Space

Proposition. *M a compact 3–manifold, $\mathcal{T}$ a triangulation of M. Then all embedded normal surfaces in M can be constructed.*

EXAMPLES. Vertex and fundamental solutions for above examples.

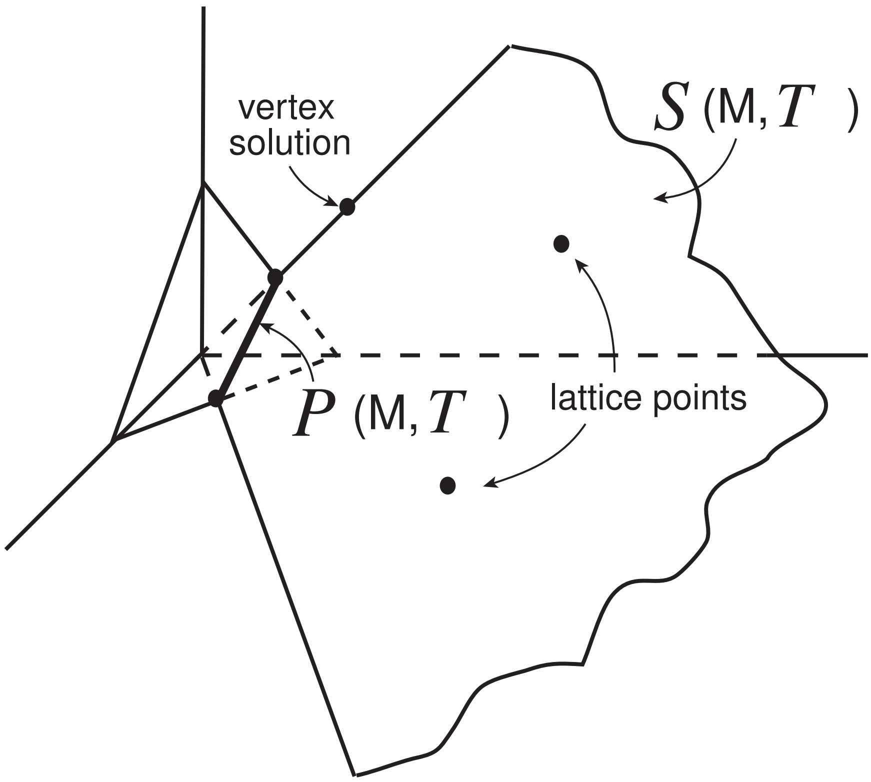
Recognition:

- We can recognize a normal disk by Euler characteristic: locally for a normal triangle $\chi(\Delta_i) = -1/2 - b_i^{(3)}/2 + \sum_{1\leq j\leq 3} 1/d_{i,j}$ and for a normal quad $\chi(\Delta_k^Q) = -1 - b_k^{(4)}/2 + \sum_{1\leq j\leq 4} 1/d_{k,j}$, thus for the normal surface $F = (x_1, \ldots, x_{4t}, y_1, \ldots, y_{3t})$

$$\chi(F) = \sum_{1\leq i\leq 4t} x_i \chi(\Delta_i) + \\ + \sum_{1\leq k\leq 3t} y_k \chi(\Delta_k^Q)$$

- We can recognize an essential disk (A s.c.c in a surface is trivial iff it bounds a disk).

- We can recognize a 2–sphere (Euler characteristic); and an essential 2–sphere **(Rubinstein, 1993)**.



- We can recognize an incompressible and $\partial-$ incompressible surface **(Haken,1961; Jaco-Oretel,1982)** and an essential surface **(Jaco-Tollefson, 1991)**

Theorem (Haken, 1961; Jaco-Oertel,1982). *If the 3–manifold M contains an essential disk, then for any triangulation $\mathcal{T}$ of M there is an essential, normal disk among the fundamental solutions* **(Jaco-Tollefson, 1991)** *among the vertex solutions).*

Corollary (Haken, 1961). *It can be decided if a knot in S^3 is trivial.*

Definition. π_1–injective, relation to incompressible

Corollary (Haken, 1961). *It can be decided if a normal surface is π_1–injective.*

Theorem (Schubert, 1964). *If the* 3*–manifold* M *contains an essential* 2*–sphere, then for any triangulation* $\mathcal{T}$ *of* M *there is an essential, normal* 2*–sphere among the fundamental solutions* **(Jaco-Tollefson, 1991)** *among the vertex solutions).*

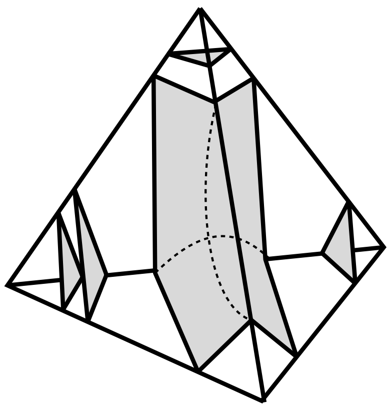
Corollary (Rubinstein, 1993). *It can be decided if a* 3*–manifold is irreducible.*

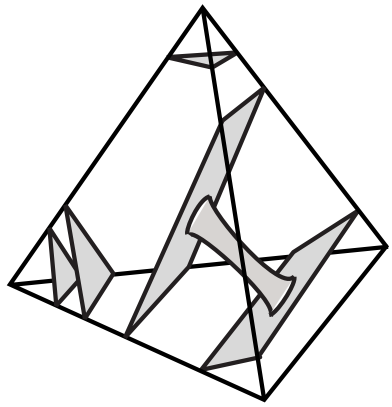
Theorem (Jaco-Oertel, 1982). *If the* 3*–manifold* M *contains an incompressible surface, then for any triangulation* $\mathcal{T}$ *of* M *there is an incompressible, normal surface among the vertex solutions.*

Corollary (Jaco-Oertel, 1982). *It can be decided if an irreducible* 3*–manifold is a Haken manifold.*

Theorem (Jaco-Reeves, 1998). *Suppose $\mathcal{T}$ is a triangulation of the 3–manifold M. If M contains a non vertex-linking, normal 2–sphere, then there is a non vertex-linking, normal 2–sphere among the vertex solutions.*

Corollary. *Suppose $\mathcal{T}$ is a triangulation of the 3–manifold M. It can be decided if $\mathcal{T}$ is 0–efficient.*



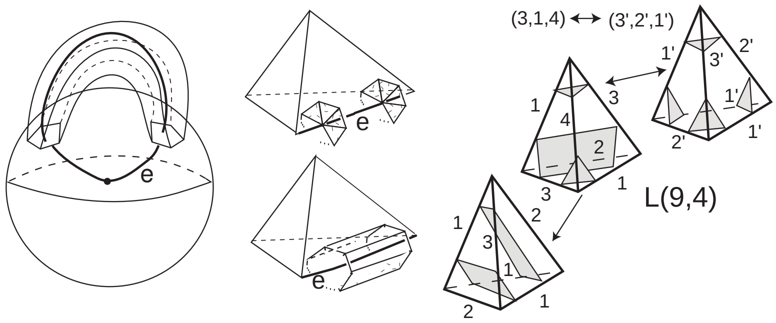


Definition. Almost normal

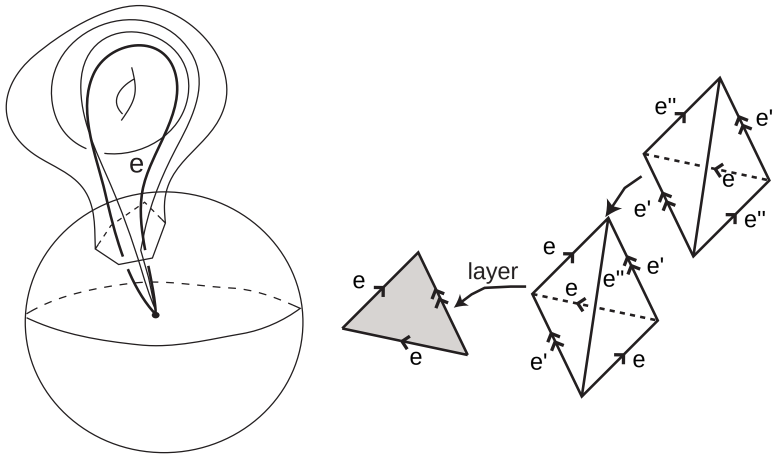
Theorem (Jaco-Reeves, 1998)**.** *Suppose M is a 3–manifold and $\mathcal{T}$ is a 0–efficient triangulation of M. If M contains an almost normal 2–sphere, then there is one among the vertex-solutions.*

Corollary. *Suppose the 3–manifold M has a 0–efficient triangulation. It can be decided if M is the 3–sphere.*

Thin Edge-linking Normal Torus



Thick Edge-linking Normal Torus



Theorem (Jaco-Reeves, 1998). *Suppose $\mathcal{T}$ is a triangulation of the 3–manifold M. If M contains a non edge-linking normal torus, then there is a non edge-linking, normal torus among the vertex solutions.*

Corollary. *Suppose $\mathcal{T}$ is a triangulation of the 3–manifold M. It can be decided if $\mathcal{T}$ is 1–efficient.*

Theorem (Jaco-Reeves, 1998)**.** *Suppose M is a 3–manifold and $\mathcal{T}$ is a 0–efficient triangulation of M. If M contains an almost normal torus, then there is one among the fundamental solutions.*

QUESTION: Must there be an almost normal torus at a vertex-solution? if the triangulation is 1–efficient, then must there be one at a vertex-solution?

Corollary. *Suppose the 3–manifold M has a 1–efficient triangulation. It can be decided if M is a lens space.*