

# III A. Normal Surface Theory: A (Brief) Survey

**Abstract:** This lecture will give a basic introduction to the theory of normal surfaces and establish some of the fundamental existence theorems.

A triangulation  $\mathcal{T}$  of a 3-manifold  $M$  determines a class of surfaces in  $M$  called **normal surfaces**.

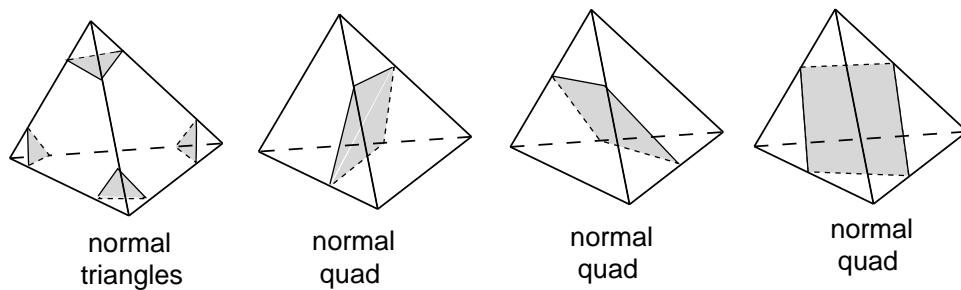


FIGURE: Normal triangles and normal quads in Normal Surface Theory.

**Definition.**  $\mathcal{T}$  a triangulation of a 3-manifold  $M$ . The surface  $S$  in  $M$  is said to be **normal** (with respect to  $\mathcal{T}$ ) if for each tetrahedron  $\Delta$  of  $\mathcal{T}$ , the components of  $S \cap \Delta$  lift to a collection of normal quads and triangles.

## **EXAMPLES:**

**A.** Meridional disks in layered triangulation of a torus.

**B.** Möbius band in one-tet-torus and Klein bottle in one-tetrahedron  $L(4,1)$ .

**C.** Family of meridional disks in Burton's example of three-tetrahedron, two-vertex solid torus.

## Existence of Normal Surfaces

**1. vertex-linking** - collection of all elementary triangles; normally isotopic to the boundary of a small regular neighborhood of the vertices.

### EXAMPLES:

**A.** Vertex-linking disk in one-tetrahedron torus.

**B.** Vertex-linking 2-spheres in one- and two-vertex, one-tetrahedron  $S^3$ .

**C.** Vertex-linking torus in two-tetrahedron ideal triangulation of Figure Eight complement.

**2. edge-linking** - basically, normally isotopic to a small regular neighborhood of an edge. This is sometimes a torus (always for one-vertex triangulations) and sometimes is not a normal surface.

### **EXAMPLES:**

**A.** Edge-linking annulus in one-tetrahedron torus.

**B.** Edge-linking torus in two-vertex, one-tetrahedron  $S^3$ ; edge-linking punctured torus about edge "2" in two-tetrahedron triangulation of solid torus extending  $\{1, 3, 4\}$ .

**C.** None about thick edge in layered triangulation of solid torus. None about edge between two vertices in two-vertex, one-tet  $S^3$ .

**3. splitting** - if  $K$  and  $K'$  are disjoint subcomplexes, then in a tetrahedron meeting both  $K$  and  $K'$ , there is a uniquely determined normal triangle or normal quadrilateral separating  $K$  from  $K'$ . The union of these normal triangles and quadrilaterals gives a normal surface separating  $K$  from  $K'$ .

## **EXAMPLES:**

**A.** Splitting torus in one-tetrahedron, two-vertex triangulation of  $S^3$ .

**B.** More generally, Heegaard surface in a two-tetrahedron triangulation determined from a Heegaard diagram (Heegaard Splitting).

**B.** Vertex-splitting surface in Burton two-vertex triangulation of  $S^3$ .

**Definition.** compression,  $\partial$ -compression

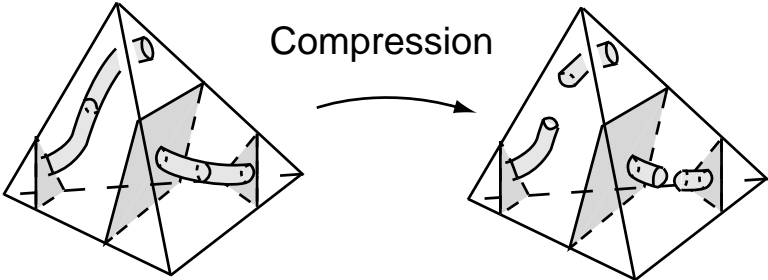
**Shrinking surfaces: normalization.** There are four basic moves in shrinking (normalizing) a properly embedded surface  $S$ : two are compressions, one an isotopy, and one a  $\partial$ -compression. If there are no normalization moves possible, the surface is said to be **stable**.

**Definition.** weight,  $wt(S)$ ; local Euler number,  $\lambda_\chi(S)$ ; and 0-weight curves,  $\sigma(S)$ ; complexity of embedded surface,  $\mathcal{C}(S) = (wt(S), \lambda_\chi(S), \sigma(S))$

The *normal moves* are:

- 1. A *compression in the interior of a tetrahedron*. This move reduces the local Euler number and does not change weight or the number of 0-weight curves of intersection of  $S$  with the faces of  $\mathcal{T}$ .

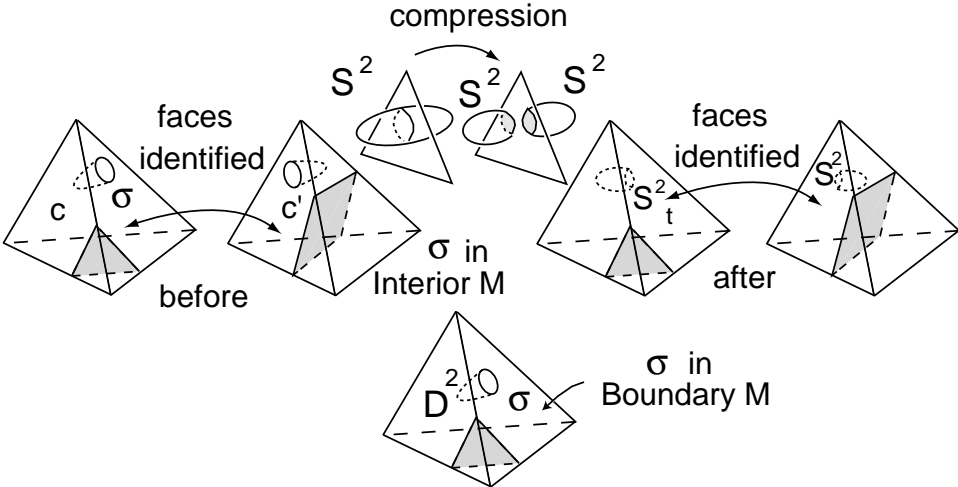
Note: Move may be an essential compression of the surface  $S$  and thereby a change of its topological type.



**FIGURE:** A compression in the interior of a cell. This move reduces the local Euler number and does not change weight or the 0-weight curves in the interior of faces of  $\mathcal{T}$ .

2. A compression eliminating 0-weight simple closed curve components from the intersection of  $S$  with the faces of  $\mathcal{T}$  in the interior of  $M$ .

These moves reduce the value  $\sigma(S)$  and do not affect the weight or the local Euler number.

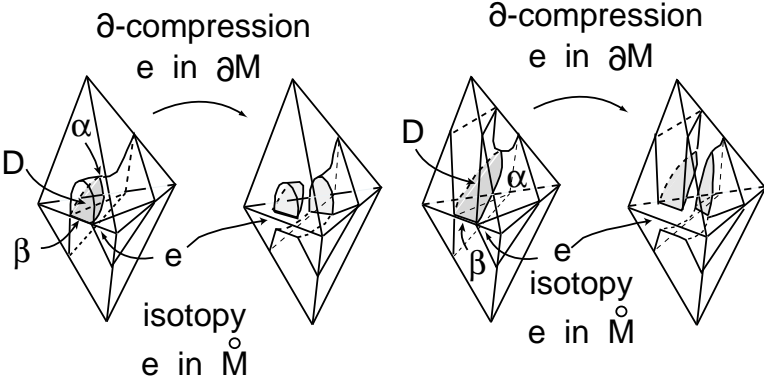


**FIGURE:** A compression removing a simple closed curve component from the intersection of  $S$  with a face in the interior of the manifold.

3. An isotopy or  $\partial$ -compression (the edge is in the  $\partial M$ ) reducing the number of times the boundary of a lift of a component meets an edge of a cell. This move reduces  $wt(S)$ .

Note: This move may increase the value  $\sigma(S)$  and the local Euler number; however, it reduces the complexity of  $S$ .

Note: A  $\partial$ -compression can change the topological type of  $S$ .



**FIGURE.** An isotopy or  $\partial$ -compression reducing the number of times a lift of a component meets an edge of a cell. This move reduces  $wt(S)$ .

- Moves can be made on any surface or surface stable.
- Moves never increase weight.

**Definition.** A sequence of these normal moves is called a *shrinking* of the surface  $S$ .  $S$  can be compressed and  $\partial$ -compressed, possibly resulting in a number of distinct components.

**Proposition.** *Suppose  $\mathcal{T}$  is a triangulation of the 3-manifold  $M$ , a properly embedded surface  $S$  in  $M$  is stable if and only if the components of  $S$  are normal in  $M$  (or, possibly, 0-weight properly embedded 2-spheres and 2-disks, each contained entirely in some tetrahedron of  $\mathcal{T}$ ).*

Note: It is possible that a surface shrinks to nothing but 0-weight components.

## Existence of Interesting Normal Surfaces:

**Definition.** incompressible surface,  $\partial$ -incompressible surface, irreducible 3-manifold,  $\partial$ -irreducible 3-manifold, essential surface

**Theorem.**  *$M$  a 3-manifold and  $M$  contains an essential surface  $S$ . Then for any triangulation  $\mathcal{T}$  of  $M$  there is an essential normal surface in  $M$  that is homeomorphic to  $S$ .*

1.  $M$  contains an essential 2-sphere, then for any triangulation there is an essential normal 2-sphere in  $M$ .
2.  $M$  contains an essential disk, then for any triangulation of  $M$  there is an essential normal disk in  $M$ .
3.  $M$  contains an incompressible and  $\partial$ -incompressible surface  $S$ , then for any triangulation of  $M$  there is an incompressible and  $\partial$ -incompressible normal surface in  $M$ , which is homeomorphic to  $S$  (isotopic to  $S$  for  $M$  irreducible and  $\partial$ -irreducible).