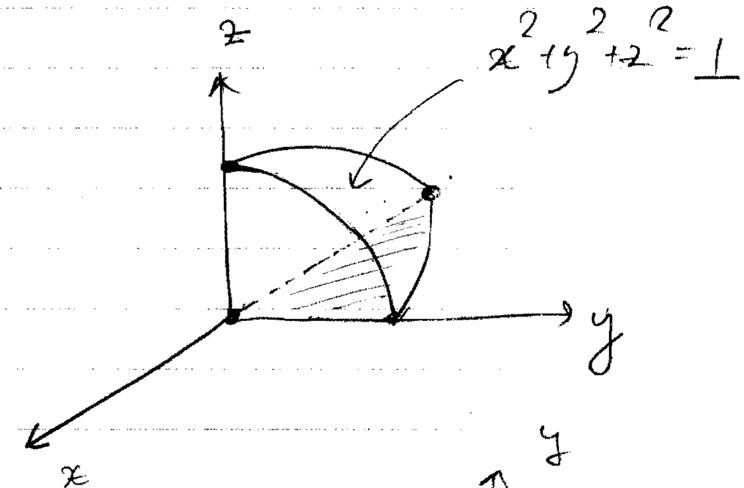
Solutions to Practice Problems for Final Exam, Spring 2009

1) $I = \int_{\frac{\pi}{2}}^{\pi} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^3 \sin\Phi \cos\Phi \, d\rho \, d\Phi \, d\theta$

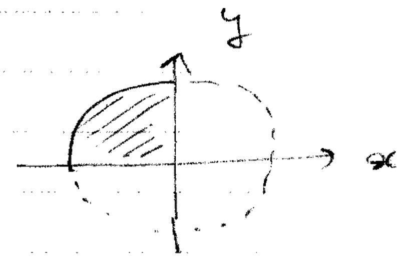
$= \iiint_E (\rho \cos\Phi)(\rho^2 \sin\Phi) \, d\rho \, d\Phi \, d\theta,$

$E = \left\{ (\rho, \Phi, \theta) \middle/ \begin{array}{l} 0 \le \rho \le 1 \\ 0 \le \Phi \le \frac{\pi}{2} \\ \frac{\pi}{2} \le \theta \le \pi \end{array} \right\}$



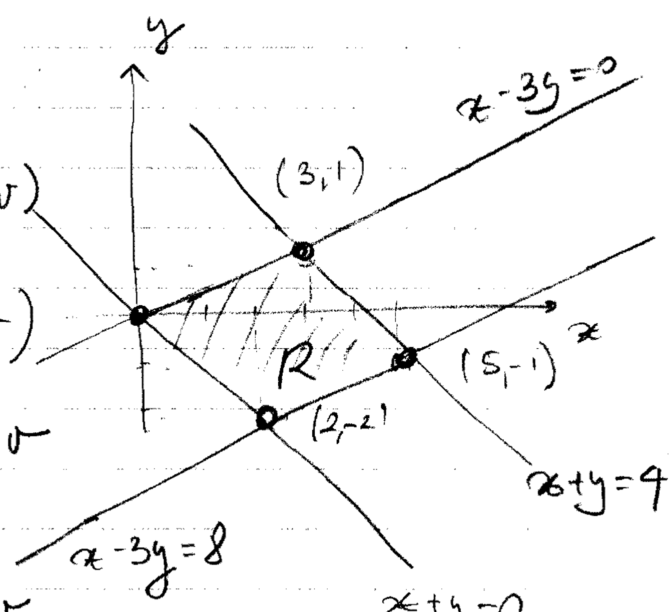
So:

$I = \int_0^1 \int_{-\sqrt{1-y^2}}^{0} \int_0^{\sqrt{1-x^2-y^2}} z \, dz \, dx \, dy$



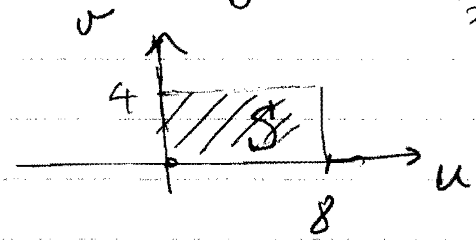
2) We use the change of variables

$T^{-1}: \begin{cases} u = x - 3y \\ v = x + y \end{cases}$, i.e. $T: \begin{cases} x = \frac{1}{4}(u + 3v) \\ y = \frac{1}{4}(-u + v) \end{cases}$



$\iint_R (x+y) \, dA = \iint_S v \, |J_T(u,v)| \, du \, dv$

$J_T(u,v) = \begin{vmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{4}$; so



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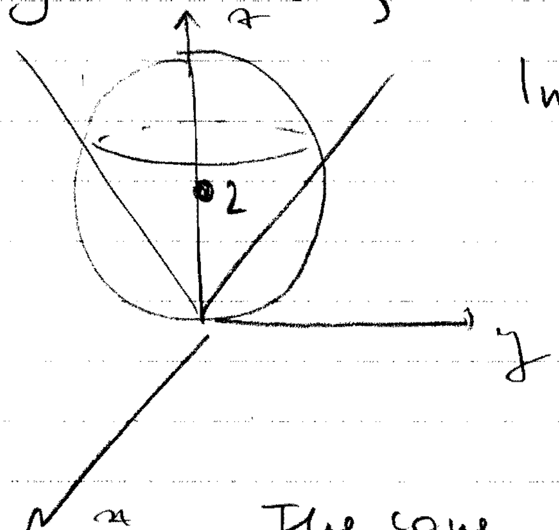
$$\iint_{S} v \left| J_T(u,v) \right| du\, dv = \int_0^4 \int_0^8 v \cdot \frac{1}{4} \, du \, dv =$$

$$= \frac{1}{4} \cdot 8 \cdot \left(\frac{1}{2} v^2\right)\Big|_0^4 = 16.$$

3) We'll use a change of variables to spherical coordinates

$x^2+y^2+z^2 = 4z$, $x^2+y^2+z^2-4z+4=0$, $x^2+y^2+(z-2)^2=0$



In spherical coordinates $x^2+y^2+z^2 = 4z$ means

$$\rho^2 = 4\rho\cos\Phi$$

$$\rho = 4\cos\Phi$$

The cone $z = \sqrt{3(x^2+y^2)}$ is written in spherical coordinates as

$$\rho\cos\Phi = \sqrt{3\rho^2\sin^2\Phi} \quad \text{ie}$$

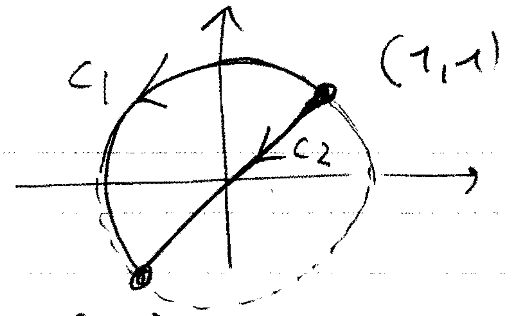
$$\cos\Phi = \sqrt{3}\sin\Phi, \quad \tan\Phi = \frac{1}{\sqrt{3}}, \quad \Phi = \frac{\pi}{6}.$$

So $\text{Volume} = \iiint_E 1 \, dV = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{4\cos\Phi} \rho^2 \sin\Phi \, d\rho \, d\Phi \, d\theta$

$$= 2\pi \int_0^{\frac{\pi}{6}} \frac{1}{3}(4\cos\Phi)^3 \sin\Phi \, d\Phi = -\frac{32\pi}{3}(\cos\Phi)^4 \Big|_0^{\frac{\pi}{6}}$$

$$= \frac{-32\pi}{3}\left(\frac{9}{16} - 1\right) = \frac{14\pi}{3}$$

4) The arc C_1 is parametrized by



$\vec{r}(t) = \langle \sqrt{2}\cos t, \sqrt{2}\sin t \rangle$,

$\frac{\pi}{4} \le t \le \frac{5\pi}{4}$,

$$W_1 = \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt =$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \langle -2\cos t \sin t, 2\cos^2 t \rangle \cdot \langle -\sqrt{2}\sin t, \sqrt{2}\cos t \rangle dt =$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 2\sqrt{2} \cos t \left(\sin^2 t + \cos^2 t \right) dt =$$

$$= 2\sqrt{2} \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos t\, dt = 2\sqrt{2} \sin t \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = -8.$$

b) The segment C_2 is parametrized by

$$\vec{r}(t) = (1-t)\langle 1,1 \rangle + t\langle -1,-1 \rangle = \langle 1-2t, 1-2t \rangle$$

$0 \le t \le 1$

$$W_2 = \int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \langle -(1-2t)^2, (1-2t)^2 \rangle \cdot \langle -2, -2 \rangle dt$$

$$= \int_0^1 0\, dt = 0.$$

c) $\vec{F}$ is not conservative on $\mathbb{R}^2$. If it was, the independence of path would hold and $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ which is not the case.

Another way to justify that $\vec{F}$ is not conservative is to note ~~that~~

$$\frac{\partial}{\partial x}(x^2) = 2x \neq -x = \frac{\partial}{\partial y}(-xy).$$

5) A parametrization for the line segment C is given by

$$\vec{r}(t) = (1-t)\langle 1,0\rangle + t\langle 2,3\rangle = \langle 1+t, 3t\rangle \qquad 0 \le t \le 1.$$

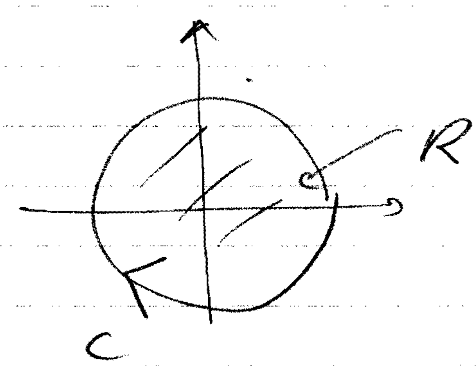
$$\int_C x^2\,ds = \int_0^1 (x(t))^2 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt =$$

$$= \int_0^1 (1+t)^2 \sqrt{1^2+3^2}\,dt = \frac{\sqrt{10}}{3}(1+t)^3\Big|_0^1 = \frac{7\sqrt{10}}{3}$$

$$\int_C x^2\,dy = \int_0^1 (x(t))^2\, y'(t)\,dt = \int_0^1 (1+t)^2 \cdot 3\,dt =$$

$$= (1+t)^3\Big|_0^1 = 7.$$

6) $$\int_C x^2y\,dx - xy^2\,dy =$$



$$= -\int_{-C} x^2y\,dx - xy^2\,dy =$$

$$= -\iint_R \left\{\frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2y)\right\} dA = \iint_R (y^2+x^2)\,dA =$$

$$= \int_0^{2\pi}\int_0^2 r^2 \cdot r\,dr\,d\theta = 2\pi\left(\frac{1}{4}r^4\right)\Big|_0^2 = 8\pi.$$

7) a) $D = \{(x,y,z) \,/\, y > 0\}$.

We need to find $f(x,y,z)$, $f: D \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial f}{\partial x} = 3x^2 \\ \dfrac{\partial f}{\partial y} = \dfrac{z^2}{y} \\ \dfrac{\partial f}{\partial z} = 2z \ln y \\ f(0,1,1) = 1 \end{cases}$$

$\Rightarrow f(x,y,z) = x^3 + g(y,z)$

So:

$$\frac{z^2}{y} = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y};$$

Hence

$$g(y,z) = z^2 \ln y + \cancel{h}\,h(z)$$

Then:

$$2z\ln y = \frac{\partial g}{\partial z} = 2z \ln y + h'(z)$$

So $h'(z) = 0$, $h(z) = k$.

Hence $f(x,y,z) = x^3 + z^2 \ln y + k$

Since $f(0,1,1) \Rightarrow k = 1$.

So $f(x,y,z) = x^3 + z^2 \ln y + 1$.

b) By the fundamental thm for line integrals (FTC)

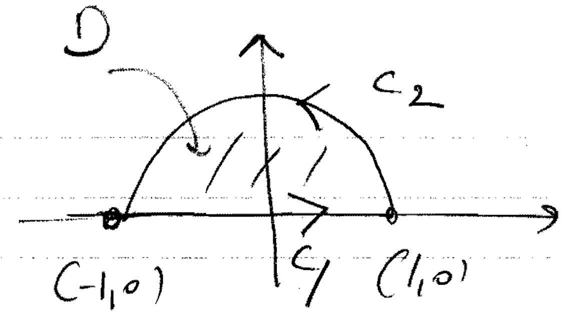
$$\int_C \vec{F} \cdot d\vec{r} = f(D) - f(A) =$$

$$= f(1,2,2) - f(2,1,3) =$$

$$= (1 + 4\ln 2 + 1) - (4 + 9\ln 1 + 1) =$$

$$= -3 + 4\ln 2.$$

8) a) The line segment C_1 is parametrized as



$\vec{r}(t) = \langle t, 0\rangle, \; -1 \le t \le 1.$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{-1}^{1} \langle 1, t^2+2t\rangle\cdot\langle 1,0\rangle\,dt = \int_{-1}^{1} 1\,dt = 2.$$

b) According to Green's Thm:

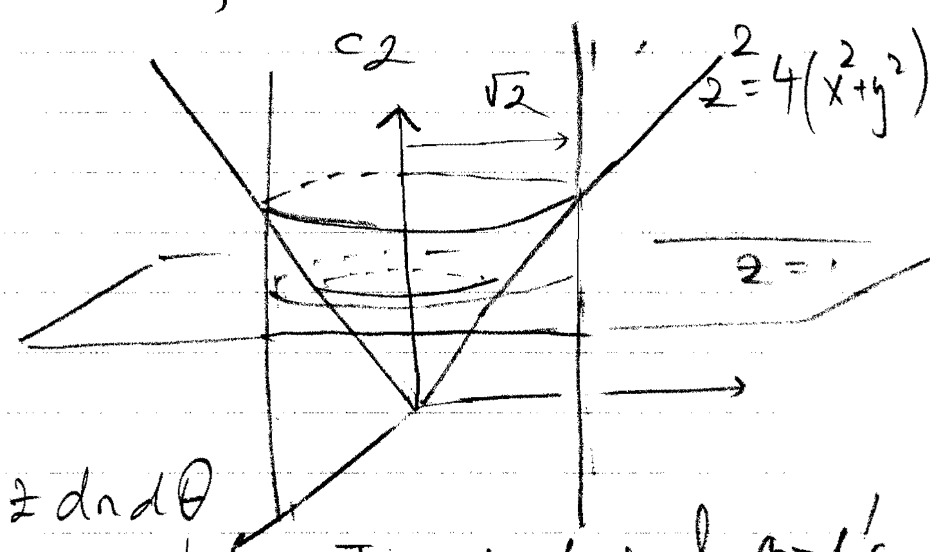
$$\left(\int_{C_1} + \int_{C_2}\right)(\vec{F}\cdot d\vec{r}) = \iint_D \left(\frac{\partial}{\partial x}(x^2+2x) - \frac{\partial}{\partial y}(2xy+1)\right)dA$$

$$= \iint_D (2x+2-2x)\,dA = 2\iint_D 1\,dA = 2\,\text{area}(D) = \pi$$

c) By a) and b):

$$2 + \int_{C_2} \vec{F}\cdot d\vec{r} = \pi, \text{ so } \int_{C_2} \vec{F}\cdot d\vec{r} = \pi - 2.$$

9) We change coordinates to cylindrical



$$\iiint_E x^2\,dV = \int_0^{2\pi}\int_{\frac{1}{2}}^{1}\int_{1}^{2r} r^2\cos^2\theta\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1+\cos(2\theta)}{2}\,d\theta \cdot \int_{\frac{1}{2}}^{\sqrt{2}} r^3(2r-1)\,dr =$$

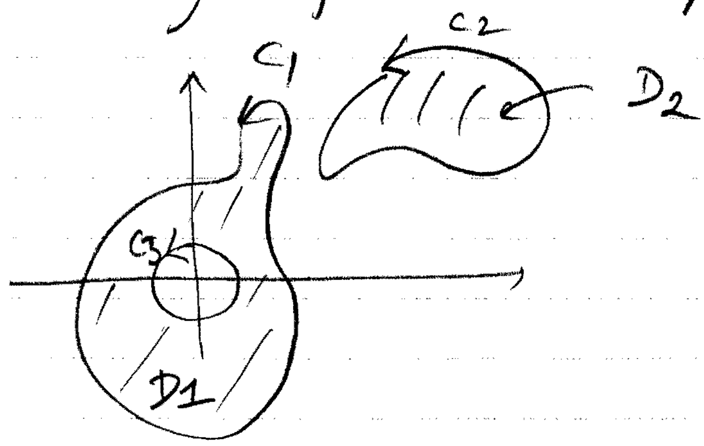
In cylindrical coord's the cone $z^2 = 4(x^2+y^2)$ is $z = 2r$ and it intersects the plane $z=1$ at $r = \frac{1}{2}$.

$$= \frac{1}{2}(2\pi)\left\{\frac{2}{5}r^5 - \frac{1}{4}r^4\right\}\Big|_{\frac{1}{2}}^{\sqrt{2}} =$$

$$= \pi\left\{\left(\frac{2}{5}4\sqrt{2} - \frac{1}{4}\cdot 4\right) - \left(\frac{2}{5}\cdot\frac{1}{32} - \frac{1}{4}\cdot\frac{1}{16}\right)\right\}$$

$$= \pi\left(\frac{8\sqrt{2}}{5} - 1 + \frac{3}{320}\right)$$

10) According to problem #33 / 17.3, $\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$



The functions $\frac{x}{x^2+y^2}$ and $\frac{-y}{x^2+y^2}$ are smooth on D_2, so by Green's theorem we have that

$$\int_{C_2} \vec{F}\cdot d\vec{n} = \iint_{D_2} 0\, dA = 0.$$

The functions $\frac{x}{x^2+y^2}$ and $\frac{-y}{x^2+y^2}$ are also smooth in D_1 (since the origin is not in D_1), so by Green's thm

$$\left(\int_{C_1} - \int_{C_3}\right) \vec{F}\cdot d\vec{n} = \iint_{D_1} 0\, dA = 0, \text{ ie}$$

$\int_{C_1} \vec{F}\cdot d\vec{n} = \int_{C_3} \vec{F}\, d\vec{n}$. Assume that C_3 is a circle of radius a around the origin, parametrized by $\vec{r}(t) = \langle a\cos t, a\sin t \rangle$, 8

$0 \le t \le 2\pi$

so $$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left\langle -\frac{a\sin t}{a^2}, \frac{a\cos t}{a^2} \right\rangle \cdot \langle -a\sin t, a\cos t \rangle dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi.$$

So: $\int_{C_1} \vec{F} \cdot d\vec{r} = 2\pi.$