Lectures 9–

GROUP COHOMOLOGY.

Let Γ be any group.

Defn By a <u>Γ- module</u> we mean an abelian group A on which Γ acts :-

So we are given a map $\Gamma \times A \longrightarrow A$ $\quad (\gamma, a) \longmapsto \gamma a$.

such that $\quad 1 \cdot a = a, \quad \gamma(a+b) = \gamma a + \gamma b, \quad \gamma_1 \gamma_2 \cdot a = \gamma_1(\gamma_2 a)$.

Let Γ- MOD be the category of Γ- modules.

Note that this is an abelian category.

$$\mathrm{Hom}_{\Gamma}(A, B) = \{ f : A \to B \mid f(\gamma a) = \gamma f(a) \}$$ – Γ-equivariant hom. from A to B.

Defn :- Let $\mathbb{Z}[\Gamma]$ be a <u>group ring</u> of Γ.

$\mathbb{Z}[\Gamma]$ is a free $\mathbb{Z}$- module on the set Γ with multiplication:-

$$\left(\sum n_\gamma \cdot \gamma\right)\left(\sum m_\gamma \cdot \gamma\right) = \sum_{\gamma, \mu} m_\gamma n_\mu \, \gamma\mu$$

($\sum n_\gamma \cdot \gamma$ is a formal $\mathbb{Z}$- linear combination of elements $\gamma \in \Gamma$ such that $n_\gamma \in \mathbb{Z}$ $\forall \gamma$ and $n_\gamma = 0$ $\forall' \gamma$ $\qquad \forall' =$ for almost all $=$ for all but finitely many

An elementary exercise:- $\quad \Gamma$ MOD $=$ $\mathbb{Z}[\Gamma]$- MOD

Any Γ- module (an abelian group with a Γ- action) is the same as a module for $\mathbb{Z}[\Gamma]$ and conversely.

Example :- The trivial Γ- module $\mathbb{Z}$:-

- $\mathbb{Z}$ is an abelian group
- Γ action on $\mathbb{Z}$ is trivial $\Rightarrow$ $\gamma \cdot n = n$ $\forall \gamma \in \Gamma$, $\forall n \in \mathbb{Z}$.

Defn. (Group cohomology).

Let Γ be any group.

Let A be a Γ-module. For any $n \geq 0$

n^{th}-cohomology group of Γ with coefficients in A is :-

$$\boxed{H^n(\Gamma, A) := \mathrm{Ext}^n_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, A).}$$

In other words, one looks at the functor

$$\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, -)$$

which is a left exact covariant functor and $H^n(\Gamma, A)$ is the n^{th}-right derived functor.

Lemma
$$\boxed{\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, A) = A^{\Gamma}}$$

$A^{\Gamma} =$ Γ-invariants of A
$= \{a \in A : \gamma a = a \ \forall \gamma \in \Gamma\}.$

If $f \in \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, A)$

then $f(1) \in A^{\Gamma}$ because : $\gamma \cdot f(1) = f(\gamma \cdot 1) = f(1) \quad \forall \gamma \in \Gamma.$

conversely, if $a \in A^{\Gamma}$, define $f : \mathbb{Z} \to A$ by $f(n) = na$. ... etc.

Therefore, $\mathrm{Hom}_{\Gamma}(\mathbb{Z}, -)$ is nothing but the functor of taking Γ-invariants.

$$H^n(\Gamma, A) = n^{th}\text{-derived functor of } (A \mapsto A^{\Gamma}).$$

~~Preclass note~~

Long exact sequence of group cohomology :-

Any short exact sequence

$$0 \to A \to B \to C \to 0$$

of Γ- modules gives a long exact sequence of abelian groups :-

$$0 \to A^{\Gamma} \to B^{\Gamma} \to C^{\Gamma} \to H^1(\Gamma, A) \to H^1(\Gamma, B) \to H^1(\Gamma, C) \to H^2(\Gamma, A) \to$$

To compute $H^n(\Gamma, A)$. :-

$$H^n(\Gamma, A) = \mathrm{Ext}^n_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, A).$$

- Start with a projective resolution of $\mathbb{Z}$, in the category of Γ-modules.

$$0 \leftarrow \mathbb{Z} \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$$

- Apply $\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(-, A)$, ~~[illegible]~~ and consider the sequence:

$$\mathrm{Hom}_{\Gamma}(P_0, A) \to \mathrm{Hom}_{\Gamma}(P_1, A) \to \cdots$$

- The cohomology of this sequence gives $H^n(\Gamma, A)$.

OR :- Start with an injective resolution of A :-

$$0 \to A \to I^0 \to I^1 \to \cdots$$

Apply $\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, -)$, which is simply taking Γ invariants, and consider

$$I^{0,\Gamma} \to I^{1,\Gamma} \to \cdots$$

Take cohomology of this sequence to get $H^n(\Gamma, A)$.

Theorem (A free resolution for $\mathbb{Z}$ in Γ-MOD).

For $n \geq 0$, define

$$C_n = \underbrace{\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma]}_{(n+1)\text{- factors}}.$$

- C_n is a free $\mathbb{Z}[\Gamma]$-module on the base $= \{(1 \otimes g_1 \otimes \cdots \otimes g_n) : g_i \in \Gamma\}$
- $\varepsilon : C_0 \longrightarrow \mathbb{Z}$ is taken as $\varepsilon(\sum n_r r) = \sum n_r$.
- $d_n : C_n \longrightarrow C_{n-1}$ is defined as :- Let $1 \otimes g_1 \otimes \cdots \otimes g_n =: (g_1, \ldots, g_n)$

$$\begin{aligned} d_n(g_1, \ldots, g_n) &= g_1(g_2, \ldots, g_n) \\ &+ \sum_{i=1}^{n-1} (-1)^i (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) \\ &+ (-1)^n (g_1, \ldots, g_{n-1}). \end{aligned}$$

Then $0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \longleftarrow \cdots$

is a free resolution of $\mathbb{Z}$ in Γ-mod.

(Proof) ---: Reading assignment, Jacobson, Basic Algebra, Vol-II, §6.9.

An alternative way to compute $H^n(\Gamma, A)$ ---

~~After~~ After all, $H^n(\Gamma, A) = H^n(\mathrm{Hom}_{\Gamma}(C_\bullet, A))$

C_m = free $\mathbb{Z}[\Gamma]$-module on the set $\underbrace{\Gamma \times \cdots \times \Gamma}_{m}$

$\therefore \mathrm{Hom}_{\Gamma}(C_m, A) \overset{\text{Bijection}}{=\!=} \text{Functions}(\Gamma \times \cdots \times \Gamma, A)$.

Defn-2

Let Γ be a group.

Let A be a Γ-module. Let $n \geq 0$

Let $C^n(\Gamma, A)$ = All functions from $\underbrace{\Gamma \times \cdots \times \Gamma}_{n} \longrightarrow A$.

with the convention that $C^0(\Gamma, A) = A$.

consider the cochain complex:-

$$C^0(\Gamma,A) \xrightarrow{d^0} C^1(\Gamma,A) \xrightarrow{d^1} \cdots \longrightarrow C^n(\Gamma,A) \xrightarrow{d^n} C^{n+1}(\Gamma,A) \longrightarrow \cdots$$

given by:-

$$\begin{aligned}(d^n f)(g_1, \ldots, g_{n+1}) &= g_1 f(g_2, \ldots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \ldots, g_n)\end{aligned}$$

Then, one can check that $d^{n+1} \circ d^n = 0$.

$H^n(\Gamma, A)$ = cohomology of this complex.

Zeroth-Cohomology :-

$H^0(\Gamma, A) =$ Kernel $(C^0(\Gamma,A) \xrightarrow{d^0} C^1(\Gamma,A))$. $= Z^0(\Gamma,A)$

Now $C^0(\Gamma,A) = A$. Let $a \in A$.

$d^0 a \in C^1(\Gamma,A)$ and is given by $d^0 a(g) = g \cdot a - a$.

$\therefore a \in A$ is in $Z^0(\Gamma,A) \iff d^0 a(g) = 0 \;\; \forall g \iff g \cdot a = a \;\; \forall g \in \Gamma$

$\iff a \in A^{\Gamma}$.

$$\therefore \quad \boxed{H^0(\Gamma, A) = A^{\Gamma}.}$$

First Cohomology group: $H^1(\Gamma, A)$

- $Z^1(\Gamma, A)$ consists of all $f : \Gamma \to A$ such that $d^1 f = 0$.

$$d^1 f(g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1)$$

$$\therefore d^1 f = 0 \iff f(g_1 g_2) = g_1 f(g_2) + f(g_1).$$

Such a function $f : \Gamma \to A$ satisfying $\boxed{f(xy) = x f(y) + f(x).}$

is called a <u>crossed homomorphism.</u>

- $B^1(\Gamma, A)$ = all elements of $C^1(\Gamma, A)$ which look like $d^0 a$ for some $a \in A = C^0(\Gamma, A)$

= all function $f : \Gamma \to A$ s.t. $\exists\, a \in A$ ~~for all~~ so that

$$f(\gamma) = \gamma . a - a.$$

Note that, if $f(x) = x.a - a$ then
$$\begin{aligned} f(xy) &= xy.a - a \\ &= x(ya - a) + xa - a \\ &= x f(y) + f(x) \end{aligned}$$

$$\Rightarrow B^1(\Gamma, A) \subset Z^1(\Gamma, A).$$

- $H^1(\Gamma, A) = Z^1(\Gamma, A) / B^1(\Gamma, A)$.

- If Γ acts trivially on A, i.e., $x.a = a \;\; \forall x \in \Gamma, \; \forall a \in A$

then $Z^1(\Gamma, A) = \mathrm{Hom}(\Gamma, A)$ & $B^1(\Gamma, A) = 0$.

$\Rightarrow H^1(\Gamma, A) = \mathrm{Hom}(\Gamma, A)$ = all homomorphisms of Γ to A.

Examples:- to do with $H^1(\Gamma, A)$. :-

① consider a short exact sequence of Γ- modules.

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

Now take invariants under Γ, which is only left-exact, to get

$$0 \to A^\Gamma \to B^\Gamma \to C^\Gamma.$$

Let us analyze what is the obstruction to the surjectivity of the map $B^\Gamma \to C^\Gamma$? <u>Answer</u>:- It is $H^1(\Gamma, A)$.

Let $c \in C^\Gamma$. Since $B \twoheadrightarrow C$, $\exists\, b \in B \ni j(b) = c$.

Then $j(\gamma b) = \gamma j(b) = \gamma . c = c \quad \forall \gamma \in \Gamma$

$\Rightarrow \quad j(\gamma b) = j(b) \quad$ or $\quad j(\gamma b - b) = 0 \quad \forall \gamma \in \Gamma$

$\Rightarrow \quad \exists!\, a \in A$ s.t $\gamma b - b = i(a)$. or $i^{-1}(\gamma b - b) \in A$.

Define $f : \Gamma \to A$ by $f(\gamma) = \gamma b - b$. (For simplicity let i be the inclusion.)

<u>check</u>:- $f \in Z^1(\Gamma, A)$ $\quad f(xy) = xyb - b = x(yb - b) + xb - b = xf(y) + f(x)$.

If we started with a different $b' \in B$ s.t. $j(b') = c$

Then we would get $f' : \Gamma \to A$ with $f'(\gamma) = \gamma . b' - b'$

Then, $(f' - f)(\gamma) = f'(\gamma) - f(\gamma) = (\gamma b' - b') - (\gamma b - b)$
$= \gamma(b' - b) - (b' - b)$

But $j(b') = j(b) \Rightarrow b' - b \in A$. Say, $b' - b = a_0$.

$\Rightarrow (f' - f)(\gamma) = \gamma a_0 - a_0 \quad \Rightarrow \quad f' - f \in B^1(\Gamma, A)$.

Hence <u>$c \in C^\Gamma$ determines a well defined class $[f] \in H^1(\Gamma, A)$.</u>

If $H^1(\Gamma, A) = (0)$ then given f as above, $\exists\, a \in A$ s.t
$f(\gamma) = \gamma . a - a \quad \Rightarrow \quad \gamma b - b = \gamma a - a \quad \Rightarrow \quad \gamma(b - a) = b - a$

$\Rightarrow b - a \in B^\Gamma$ and $j(b - a) = jb - 0 = c$.

$\Rightarrow$ The map $B^\Gamma \to C^\Gamma$ is surjective.

② Hilbert Theorem 90 :-

The classical Hilbert Theorem 90 says this :-

Theorem 1

Let E/F be a cyclic Galois extn, say $Gal(E/F) = \{1, \sigma, \sigma^2, \dots, \sigma^{n-1}\}$.
If $x \in E^*$ is such that $N_{E/F}(x) = 1$ then $\exists\, y \in E^*$ such that $x = y/\sigma(y)$.

In other words $Ker(N_{E/F}) = \{ x \in E^* : x = y/\sigma(y)$ for some $y \in E^* \}$.

Let $G = Gal(E/F)$, then E^* is a G-module.

The following is a cohomological version of Hilbert 90 :-

Theorem 2

Let E/F be any Galois extn. Let $G = Gal(E/F)$.
Then $H^1(G, E^*) = \{1\}$.

When E/F is cyclic, there is a particularly nice resolution of $\mathbb{Z}$ using free $\mathbb{Z}[G]$-modules. From this resolution one can see that $Z^1(G, E^*) = Ker(N_{E/F})$. and $B^1(G, E^*) = \{ y/\sigma(y) \}$.

If $G = \{1, \sigma, \dots, \sigma^{n-1}\}$. that nice resolutions looks like this :-

$$\cdots \longrightarrow \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \xrightarrow{1+\sigma+\cdots+\sigma^{n-1}} \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

Second Cohomology group:-

$$Z^2(\Gamma, A) = \{ f: \Gamma\times\Gamma \longrightarrow A \mid g_1 f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1,g_2) = 0 \ \forall g_1, g_2, g_3 \in \Gamma \}.$$

$$B^2(\Gamma, A) = \{ f: \Gamma\times\Gamma \longrightarrow A \mid \exists \beta: \Gamma \longrightarrow A \text{ so that } f = d'\beta \text{ or } f(x,y) = x\beta(y) - \beta(xy) + \beta(x) \}.$$

Example:- When does a projective representation of a group G come from an honest-to-goodness representation? When is there a dotted arrow:-

$$\begin{tikzcd} & G \arrow[d, "\pi"] \arrow[dl, dashed, "??"'] \\ GL_n(\mathbb{C}) \arrow[r] & PGL_n(\mathbb{C}) \end{tikzcd}$$

Answer:- $\boxed{H^2(G, \mathbb{C}^*) = \{1\} \implies \text{Dotted arrow exists!}}$

Proof:- Consider the diagram:-

$$\begin{tikzcd} & & & G \arrow[d, "\pi"] \\ 1 \arrow[r] & \mathbb{C}^* \arrow[r] & GL_n(\mathbb{C}) \arrow[r, "p"] & PGL_n(\mathbb{C}) \arrow[r] \arrow[l, bend left, "s"] & 1 \end{tikzcd}$$

with s being any set-theoretic section, i.e., $s: PGL_n(\mathbb{C}) \longrightarrow GL_n(\mathbb{C})$ is just a function with $p\circ s = 1_{PGL_n(\mathbb{C})}$.

Let $f(x) = s(\pi(x))$. So $f: G \longrightarrow GL_n(\mathbb{C})$.

So f is a candidate for the dotted arrow. But f need not be a homomorphism. What is the obstruction? Define $\alpha: G\times G \longrightarrow GL_n$

$$\alpha(x,y) = f(xy) f(y)^{-1} f(x)^{-1}.$$

$$p(\alpha(x,y)) = p(s(\pi(xy))\, s(\pi(y))^{-1} s(\pi(x))^{-1}) = \cdots = 1.$$

Hence, $\alpha(x,y) \in \mathbb{C}^*$. , i.e., $\alpha : G \times G \longrightarrow \mathbb{C}^*$.

Check:- $\alpha \in Z^2(G, \mathbb{C}^*)$ with the G- action on $\mathbb{C}^*$ being the trivial action.

So check that $\alpha(y,z) \cdot \alpha(x,yz) = \alpha(xy,z)\,\alpha(x,y) \quad \forall\, x,y,z \in G$.

Suppose, $H^2(G, \mathbb{C}^*) = \{1\}$.

$\Rightarrow \quad \alpha \in Z^2(G, \mathbb{C}^*) = B^2(G, \mathbb{C}^*)$

$\Rightarrow \quad \exists\, \beta : G \longrightarrow \mathbb{C}^*$ such that $\alpha(x,y) = \dfrac{x\beta(y)\cdot\beta(x)}{\beta(xy)} = \beta(y)\beta(x)\beta(xy)^{-1}$

~~Also consider the~~ Now you modify $f : G \longrightarrow GL_n(\mathbb{C})$ by this map β:-

Define $\tilde{f} : G \longrightarrow GL_n(\mathbb{C})$ by

$$\tilde{f}(x) = f(x)\,\beta(x).$$

$$\tilde{f}(xy)\,\tilde{f}(y)^{-1}\,\tilde{f}(x)^{-1} = f(xy)\,f(y)^{-1}\,f(x)^{-1}\,\beta(xy)\,\beta(y)^{-1}\,\beta(x)^{-1}$$
$$= \alpha(x,y)\cdot\beta(xy)\,\beta(y)^{-1}\,\beta(x)^{-1} = 1.$$

$\Rightarrow \quad \tilde{f}$ is a homomorphism.

Further, $(p \circ \tilde{f})(g) = p(f(g)\,\beta(g)) = \cancel{p(f(g))}\ \underbrace{p(s(\pi(g)))}_{\pi(g)}\ \underbrace{p(\beta(g))}_{1}$
$$= \pi(g)$$

$\Rightarrow \quad \tilde{f}$ is the required dotted arrow!

Notes:-

$H^2(G, \mathbb{C}^*)$ is called the group of <u>Schur Multipliers</u> for G.

<u>Example</u> (Brauer group of a field F).

Let F be any field.

Let $G = Gal(\bar{F}/F)$.

Then $\bar{F}^*$ is a G-module.

We have already seen $H^1(G, \bar{F}^*) = \{1\}$ which is a generalization of the classical Hilbert 90. The second cohomology is much more nontrivial!

Brauer group of F $=$ $Br(F)$ $:=$ $H^2(G, \bar{F}^*)$.

$Br(F)$ classifies all finite dimensional central division algebras over F.

Examples :-

(i) $Br(\mathbb{R}) \cong \mathbb{Z}/2$ (Nontrivial element is represented by the Hamiltonian algebra.)

(ii) $Br(\mathbb{C}) = \{0\}$ ($\nexists$ nontrivial division algebras over an alg. closed field.)

(iii) $Br(\mathbb{F}_q) = \{0\}$ (Classical Wedderburn theorem)

(iv) $Br(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$ — (Nontrivial theorem of local class field theory.)

Homology of groups:-

Let Γ be a group and A any Γ-module.

$$\boxed{H_n(\Gamma, A) := \mathrm{Tor}_n^{\Gamma}(\mathbb{Z}, A).}$$

Recall:- $\mathrm{Tor}_n^{\Gamma}(\mathbb{Z}, A) = L_n(\mathbb{Z} \otimes_{\Gamma} -)(A)$.

Let us look at $H_0(\Gamma, A)$:-

$$H_0(\Gamma, A) = \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} A.$$

consider the exact sequence $0 \to I_\Gamma \to \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$.

where I_Γ is the kernel ideal of the augmentation homomorphism ε.

It is easy to see that I_Γ = ideal generated by $\{\gamma - 1 : \gamma \in \Gamma\}$.

Tensoring the sequence by A, i.e., apply $- \otimes_{\mathbb{Z}[\Gamma]} A$ to get:-

$$\begin{array}{ccccccc} I_\Gamma \otimes_{\mathbb{Z}[\Gamma]} A & \longrightarrow & \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma]} A & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} A & \longrightarrow & 0 \\ \wr\wr & & \wr\wr & & & & \\ I_\Gamma \cdot A & \hookrightarrow & A & & & & \end{array}$$

$$\Rightarrow \quad \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} A \cong A / I_\Gamma A.$$

$$\boxed{H_0(\Gamma, A) = A / I_\Gamma A} = \text{maximal quotient of } A \text{ on which } \Gamma \text{ acts trivially}$$

Exercise:- $\boxed{H_1(\Gamma, \mathbb{Z}) = \Gamma / [\Gamma, \Gamma].}$

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