Lecture-4

Resolutions & Homotopy.

§1 Homotopy :-

Let $(M_\bullet, d_\bullet)$ and $(N_\bullet, d_\bullet)$ be complexes.

Let $\alpha, \beta : M_\bullet \longrightarrow N_\bullet$ be homomorphism of complexes.

We say α is <u>homotopic to β</u>. if $\exists$ sequence of homomorphisms

$s_i : M_i \longrightarrow N_{i+1}$ such that

$$\alpha_i - \beta_i = d_{i+1} s_i + s_{i-1} d_i$$

$$\begin{tikzcd}
\cdots & M_{i-1} \arrow[l] \arrow[d, shift right, "\alpha_{i-1}"'] \arrow[d, shift left, "\beta_{i-1}"] \arrow[dr, "s_{i-1}"] & M_i \arrow[l, "d_i"'] \arrow[d, shift right, "\alpha_i"'] \arrow[d, shift left, "\beta_i"] \arrow[dr, "s_i"] & M_{i+1} \arrow[l] \arrow[d] & \arrow[l] \\
\cdots & N_{i-1} \arrow[l] & N_i \arrow[l, "d_i"] & N_{i+1} \arrow[l, "d_{i+1}"] & \arrow[l]
\end{tikzcd}$$

Theorem :- $\boxed{\text{Homotopic maps induce identical maps in homology.}}$

If $\alpha, \beta : M_\bullet \longrightarrow N_\bullet$ are homotopic then $\tilde{\alpha}_i = \tilde{\beta}_i : H_i(M_\bullet) \longrightarrow H_i(N_\bullet)$

Pf :- $[z_i] \in H_i(M_\bullet)$, i.e., $z_i \in Z_i(M_\bullet)$ & $[z_i] = z_i + B_i \in H_i(M_\bullet)$

Then
$$\begin{aligned}
\tilde{\alpha}_i[z_i] = \alpha_i z_i + B_i &= (\beta_i + d_{i+1} s_i + s_{i-1} d_i) z_i + B_i \\
&= \beta_i z_i + \underbrace{d_{i+1} s_i z_i}_{\in B_i} + s_{i-1} \underbrace{d_i z_i}_{=0} + B_i \\
&= \beta_i z_i + B_i = \tilde{\beta}_i([z_i])
\end{aligned}$$

Application : In computing homology we will often be working with a convenient choice of a complex (actually a resolution). After carrying out the computation we will have to check that the answer is independent of the choice of resolution. It turns out that two different resolutions are homotopic and so give the same maps in homology!

§2 Resolutions.

Let M be an R-module.

Definitions :-

① A <u>complex over M</u> is a complex $C_\circ = (C_\circ, d_\circ)$ and a homomorphism $\varepsilon : C_0 \longrightarrow M$ called <u>augmentation</u> such that $\varepsilon d_1 = 0$

So we a sequence

$$\longrightarrow C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

(The composition of two successive hom. is 0.)

② The complex $(C_\circ, \varepsilon)$ over M is called a <u>resolution</u> if the above sequence is exact.

③ A <u>projective resolution</u> is a resolution $(C_\circ, \varepsilon)$ with each C_i being a projective module.

④ A <u>free resolution</u> ... each C_i is free.

Example :-

① Let $R = \mathbb{Z}$, $R\text{-MOD} = AB$.
Let $M = \mathbb{Z}/n\mathbb{Z}$. Then $0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$ is a free resolution of $\mathbb{Z}/n\mathbb{Z}$.

② Note that ① can be generalized to the case when R is a P.I.D., because a submodule of a free module over a PID is also free.
i.e., if M is a module over R, with R a PID, then $\exists$ free resolution

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

Such a resolution is said to have length 2.

Proposition:

Let R be any ring. $(1 \in R)$

Let M be an R-module.

Then M has a free resolution: $\cdots \to F_1 \to F_0 \to M \to 0$

Since every free module is projective, every module has a projective resolution.

Proof: Let F_0 = free R-module on the "set" M.

Then is a canonical map $F_0 \xrightarrow{\varepsilon} M$.

Let $K_0 = \operatorname{Ker}(\varepsilon)$. $\exists$ free module F_1 and a canonical $F_1 \twoheadrightarrow K_0$

Define $F_1 \to F_0$ by
$$\begin{tikzcd} F_1 \arrow[r] \arrow[dr, two heads] & F_0 \\ & K_0 \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} F_1 \arrow[r, "d_1"] \arrow[dr, two heads] & F_0 \arrow[r, two heads, "\varepsilon"] & M \arrow[r] & 0 \\ & K_0 \arrow[u] & & \end{tikzcd}$$

Let $K_1 = \operatorname{Ker}(d_1)$ .... etc ....

Theorem

A homomorphism of modules $f: M \to N$ can be "lifted" to a homomorphism of projective resolutions of M and N. The lifted homomorphism is canonical up to homotopy

Let $f: M \to N$ be a hom. of R-modules.

Let $(P_\bullet, \varepsilon)$ be a projective resolution of M.

Let $(Q_\bullet, \eta)$ " " " N.

The theorem states that $\exists\ \alpha: P_\bullet \to Q_\bullet$ such that the diagram ~~commutes~~

$$\begin{tikzcd} \arrow[r] & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\alpha_{n+1}"] & P_n \arrow[r, "d_n"] \arrow[d, "\alpha_n"] & P_{n-1} \cdots \arrow[r, "d_2"] \arrow[d, "\alpha_{n-1}"] & P_1 \arrow[r, "d_1"] \arrow[d, "\alpha_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\alpha_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\ & Q_{n+1} \arrow[r, "\delta_{n+1}"] & Q_n \arrow[r, "\delta_n"] & Q_{n-1} \cdots \arrow[r, "\delta_2"] & Q_1 \arrow[r, "\delta_1"] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0 \end{tikzcd}$$

commutes.

And, if $\beta: P_\bullet \to Q_\bullet$ is another such map then $\alpha \sim \beta$.

"<u>Proof</u>":- P_0 being projective, gives α_0.

$$\begin{tikzcd} & P_0 \arrow[dl, dashed, "\alpha_0"'] \arrow[d] \arrow[r] & M \arrow[dl] \\ Q_0 \arrow[r, two heads] & N & \end{tikzcd}$$

Supposing we have constructed up to α_n.

Consider

$$\begin{tikzcd} P_{n+1} \arrow[r, "d_{n+1}"] & P_n \arrow[r] \arrow[d, "\alpha_n"] & P_{n-1} \arrow[d, "\alpha_{n-1}"] \\ Q_{n+1} \arrow[r] & Q_n \arrow[r] & Q_{n-1} \end{tikzcd}$$

The map $P_{n+1} \longrightarrow Q_n$ actually lands inside image of Q_{n+1}. (why?)

So we are looking at

$$\begin{tikzcd} & P_{n+1} \arrow[dl, dashed, "\alpha_{n+1}"'] \arrow[d] \\ Q_{n+1} \arrow[r, two heads, "\delta_{n+1}"] & \delta_{n+1}(Q_{n+1}) \end{tikzcd}$$

Projectivity of P_{n+1} gives α_{n+1}

So we have lifted f to a morphism $\alpha_\cdot : P_\cdot \longrightarrow Q_\cdot$

Now suppose we have another $\beta_\cdot : P_\cdot \longrightarrow Q_\cdot$ Let $\gamma_\cdot = \alpha_\cdot - \beta_\cdot$

$\therefore \quad f \circ \varepsilon = \eta \circ \alpha_0 = \eta \cdot \beta_0 \implies \eta \cdot \gamma_0 = 0.$

$\implies \mathrm{Im}(\gamma_0) \subset \mathrm{Ker}(\eta) = \mathrm{Im}(\delta_1)$

$\implies \exists\, s_0 : P_0 \longrightarrow Q_1$ by projectivity of P_0.

$$\begin{tikzcd} & P_0 \arrow[dl, dashed, "s_0"'] \arrow[d, "\gamma_0"] & & \\ Q_1 \arrow[r, "\delta_1"] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0 \end{tikzcd}$$

Suppose we have constructed $s_0, \ldots, s_{n-1}$. We can get s_n as follows:- $0 \le i \le n-1$.

$\gamma_i = s_{i-1} \cdot d_i + \delta_{i+1} \cdot$

$$\begin{tikzcd} \arrow[r] & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d] & P_n \arrow[r, "d_n"] \arrow[d, "\gamma_n"] \arrow[dl, dashed] & P_{n-1} \arrow[r] \arrow[d, "\gamma_{n-1}"] \arrow[dl, "s_{n-1}"'] & P_{n-2} \arrow[r] \arrow[d, "\gamma_{n-2}"] \arrow[dl, "s_{n-2}"'] & {} \\ \arrow[r] & Q_{n+1} \arrow[r, "\delta_{n+1}"] & Q_n \arrow[r, "\delta_n"] & Q_{n-1} \arrow[r] & Q_{n-2} \arrow[r] & {} \end{tikzcd}$$

Consider $\gamma_n - s_{n-1} \cdot d_n : P_n \longrightarrow Q_n$.

Then $\delta_n(\gamma_n - s_{n-1} d_n) = \delta_n \gamma_n - \delta_n s_{n-1} d_n = \gamma_{n-1} d_n - \delta_n s_{n-1} d_n.$

$= (s_{n-2} d_{n-1} + \delta_n \cdot s_{n-1}) d_n - \delta_n s_{n-1} d_n = 0.$

$\implies \mathrm{Im}(\gamma_n - s_{n-1} d_n) \subset \mathrm{Ker}(\delta_n) = \mathrm{Im}(\delta_{n+1})$. Projectivity of P_n gives s_n.... etc..

All the above statements about projective resolutions can be dualized.

Definition :-

① A complex under M is a cochain complex (A°, d°) and a homomorphism $M \xrightarrow{\eta} A^0$ such that $d^0 \circ \eta = 0$. , i.e., we have a sequence

$$0 \to M \xrightarrow{\eta} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \to \cdots \to A^n \xrightarrow{d^n} A^{n+1} \to \cdots$$

such that "Two steps $= 0$".

② A complex under M is called a coresolution if the sequence is exact.
(In practice, people do not distinguish between a coresolution & resolution.

③ An injective resolution of a module M is a coresolution

$$0 \to M \xrightarrow{\eta} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots \to I^n \xrightarrow{d^n} I^{n+1} \to \cdots$$

such that each I^j is an injective module.

Theorems (Basic facts of injective resolutions)

① Every module has an injective resolution.

② Any homomorphism of modules $f: M \to N$ can be lifted to a homomorphism of their injective resolutions. ; ~~the lifts~~ two such lifts are homotopic.

Proof :-

① First prove some facts about abelian groups.

- An injective abelian group $\equiv$ A divisible abelian group.

- If A is an abelian group then we can embed A into an injective abelian group :- Let $A^\vee = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$.
Let F be a free group such that $F \to A^\vee \to 0$.
Dualize to get $A^{\vee\vee} \hookrightarrow F^\vee$. Now $A \hookrightarrow A^{\vee\vee}$
$\therefore A \hookrightarrow F^\vee$ and dual of free is injective.

Now, given an R-module M, let M^+ be the underlying abelian group.

Let I be an injective abelian group such that $M^+ \hookrightarrow I$.

Then

$$M \simeq \mathrm{Hom}_R(R, M) \subset \mathrm{Hom}_{\mathbb{Z}}(R, M^+) \subset \mathrm{Hom}_{\mathbb{Z}}(R, I)$$

check that $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ is an injective R-module.

Once every module can be embedded into an injective module, we get an injective resolution as:-

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, hook] & I^0 \arrow[r] \arrow[dr] & I^1 \arrow[r] & \cdots \arrow[r] & I^n \arrow[rr] \arrow[dr] & & I^{n+1} \\
 & & & C^0 \arrow[u] & & & C^n \arrow[ur] &
\end{tikzcd}$$

where $C^0 = \mathrm{coker}(M \hookrightarrow I^0)$, $C^n = \mathrm{coker}(I^{n-1} \hookrightarrow I^n)$ ... etc ...

Proof of ② is exactly like the corresponding statement for projective resolutions. Just reverse all arrows!

You will be repeatedly using the fact :-

I is injective $\iff$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, hook] \arrow[d] & B \arrow[dl, dashed] \\
 & I &
\end{tikzcd}$$
