

§1 Homotopy :-

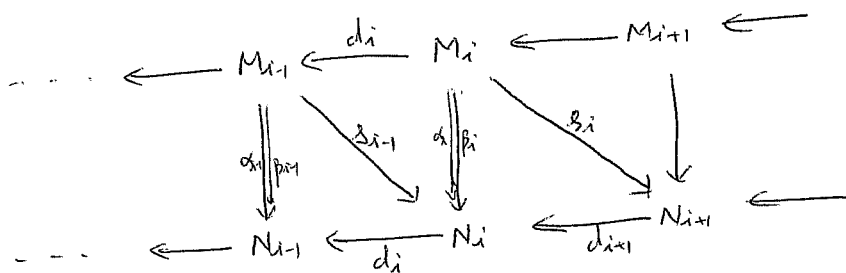
Let (M_0, d_0) and (N_0, d_0) be complexes.

Let $\alpha, \beta : M_0 \rightarrow N_0$ be homomorphism of complexes.

We say α is homotopic to β if \exists sequence of homomorphisms

$s_i : M_i \rightarrow N_{i+1}$ such that

$$d_i \alpha - \beta d_i = d_{i+1} s_i + s_{i-1} d_i$$



Theorem :-

Homotopic maps induce identical maps in homology.

If $\alpha, \beta : M_0 \rightarrow N_0$ are homotopic then $\tilde{\alpha}_i = \tilde{\beta}_i : H_i(M_0) \rightarrow H_i(N_0)$

PF:- $[z_i] \in H_i(M_0)$, i.e., $z_i \in Z_i(M_0)$ & $[z_i] = z_i + B_i \in H_i(M_0)$

$$\begin{aligned} \text{Then } \tilde{\alpha}_i [z_i] &= d_i z_i + B_i = (\beta_i + d_{i+1} s_i + s_{i-1} d_i) z_i + B_i \\ &= \beta_i z_i + \underbrace{d_{i+1} s_i z_i}_{\cap B_i} + s_{i-1} \underbrace{d_i z_i}_0 + B_i \\ &= \beta_i z_i + B_i = \tilde{\beta}_i ([z_i]) \end{aligned}$$

Application : In computing homology we will often be working with a convenient choice of a complex (actually a resolution). After carrying out the computation we will have to check that the answer is independent of the choice of resolution. It turns out that two different resolutions ~~are~~ are homotopic and so give the same maps in homology!

§2 Resolutions.

Let M be an R -module.

Definitions:-

- ① A complex over M is a complex $C_0 = (C_0, d_0)$ and a homomorphism $\epsilon: C_0 \rightarrow M$ called augmentation such that $\epsilon d_1 = 0$

So we a sequence

$$\cdots \rightarrow C_n \xrightarrow{d_n} C_{n-1} \rightarrow \cdots \rightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} M \rightarrow 0.$$

(The composition of two successive hom. is 0.)

- ② The complex (C_0, ϵ) over M is called a resolution if the above sequence is exact.
- ③ A projective resolution is a resolution (C_0, ϵ) with each C_i being a projective module.
- ④ A free resolution ... each C_i is free.

Example:-

- ① Let $R = \mathbb{Z}$, $R\text{-MOD} = AB$.

Let $M = \mathbb{Z}/n\mathbb{Z}$.

Then

$$0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$$

is a free resolution of $\mathbb{Z}/n\mathbb{Z}$.

- ② Note that ① can be generalized to the case when R is a P.I.D., because a submodule of a free module over a PID is also free.
- i.e., if M is a module over R , with R a PID, then \exists free resolution

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

Such a resolution is said to have length 2.

Proposition:

Let R be any ring. ($1 \in R$)

Let M be an R -module.

Then M has a free resolution:

$$\dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

since every free module is projective, every module has a projective resolution.

Proof:

Let $F_0 =$ free R -module on the "set" M .

Then is a canonical map $F_0 \xrightarrow{\epsilon} M$.

Let $K_0 = \text{Ker}(\epsilon)$. \exists free module F_1 and a canonical $F_1 \twoheadrightarrow K_0$

Define $F_1 \rightarrow F_0$ by $F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$

$$\begin{array}{ccccc} F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{\epsilon} & M \rightarrow 0 \\ & \searrow & \nearrow & & \\ & & K_0 & & \end{array}$$

Let $K_1 = \text{Ker}(d_1)$... etc ...

Theorem

A homomorphism of modules $f: M \rightarrow N$ can be "lifted" to a homomorphism of projective resolutions of M and N . The lifted homomorphism is canonical up to homotopy

Let $f: M \rightarrow N$ be a hom. of R -modules.

Let (P_\bullet, ϵ) be a projective resolution of M .

Let (Q_\bullet, η) " " " " N .

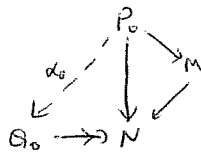
The theorem states that $\exists \alpha: P_\bullet \rightarrow Q_\bullet$ such that the diagram ~~is~~ commutes.

$$\begin{array}{ccccccccccc} \rightarrow P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} & \dots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \rightarrow 0 \\ & \downarrow d_{n+1} & \downarrow d_n & \downarrow d_{n-1} & & & & \downarrow d_1 & & \downarrow d_0 & & \downarrow f & \\ \rightarrow Q_{n+1} & \xrightarrow{s_{n+1}} & Q_n & \xrightarrow{s_n} & Q_{n-1} & \dots & \xrightarrow{s_2} & Q_1 & \xrightarrow{s_1} & Q_0 & \xrightarrow{\eta} & N & \rightarrow 0 \end{array}$$

And, if $\beta: P_\bullet \rightarrow Q_\bullet$ is another such map then $\alpha \sim \beta$.

"Proof" :-

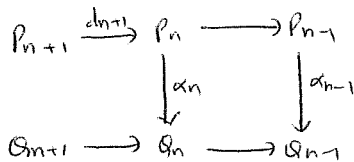
P_0 being projective,



gives α_0 .

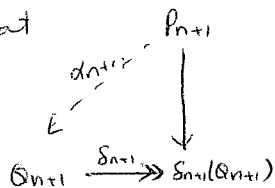
Supposing we have constructed up to α_n .

considers



The map $P_{n+1} \rightarrow Q_n$ actually lands inside image of Q_{n+1} . (why?)

So we are looking at



Projectivity of P_{n+1} gives α_{n+1}

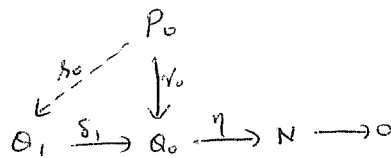
so we have lifted f to a morphism $\alpha_0 : P_0 \rightarrow Q_0$.

Now suppose we have another $\beta_0 : P_0 \rightarrow Q_0$. Let $\gamma_0 = \alpha_0 - \beta_0$

$$\therefore f \circ \varepsilon = \eta \circ d_0 = \eta \circ \beta_0 \Rightarrow \eta \circ \gamma_0 = 0$$

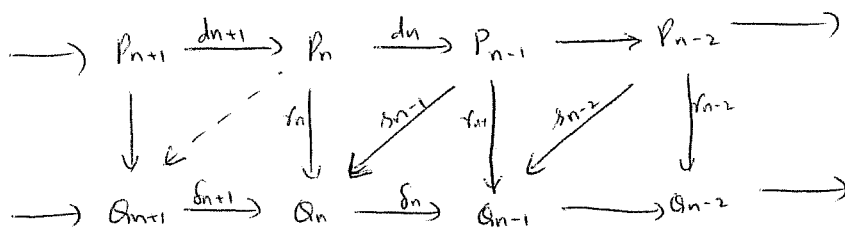
$$\Rightarrow \text{Im}(\gamma_0) \subset \text{Ker}(\eta) = \text{Im}(\delta_1)$$

$$\Rightarrow \exists \delta_0 : P_0 \rightarrow Q_1 \text{ by projectivity of } P_0.$$



Suppose we have constructed $\delta_0, \dots, \delta_{n-1}$. We can get δ_n as follows:- $0 \leq i \leq n-1$.

$$\gamma_i = \delta_{i-1} \cdot d_i + \delta_{i+1}$$



consider $\gamma_n - \delta_{n+1} \cdot d_n : P_n \rightarrow Q_n$.

$$\begin{aligned} \text{Then } \delta_n(\gamma_n - \delta_{n+1} \cdot d_n) &= \delta_n \gamma_n - \delta_n \delta_{n+1} \cdot d_n = \gamma_n \cdot d_n - \delta_n \delta_{n+1} \cdot d_n \\ &= (\delta_{n-2} \cdot d_{n-1} + \delta_n \cdot \delta_{n-1}) \cdot d_n - \delta_n \delta_{n+1} \cdot d_n = 0. \end{aligned}$$

$$\Rightarrow \text{Im}(\gamma_n - \delta_{n+1} \cdot d_n) \subset \text{Ker}(d_n) = \text{Im}(\delta_{n+1}). \text{ Projectivity of } P_n \text{ gives } \delta_n \dots \text{ etc.}$$

All the above statements about projective resolutions can be dualized.

Definition :-

① A complex under M is a cochain complex (A^\bullet, d^\bullet) and a homomorphism $M \xrightarrow{\eta} A_0^\bullet$ such that $d^0 \circ \eta = 0$, i.e., we have a sequence

$$0 \rightarrow M \xrightarrow{\eta} A_0^\bullet \xrightarrow{d^0} A_1^\bullet \xrightarrow{d^1} A_2^\bullet \rightarrow \dots \rightarrow A^n \xrightarrow{d^n} A^{n+1} \rightarrow \dots$$

such that "Two steps = 0".

② A complex under M is called a coresolution if the sequence is exact. (In practice, people do not distinguish between a coresolution & resolution.)

③ An injective resolution of a module M is a coresolution

$$0 \rightarrow M \xrightarrow{\eta} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \rightarrow \dots \rightarrow I^n \xrightarrow{d^n} I^{n+1} \rightarrow \dots$$

such that each I^i is an injective module.

Theorems (Basic facts of injective resolutions)

① Every module has an injective resolution.

② Any homomorphism of modules $f: M \rightarrow N$ can be lifted to a homomorphism of their injective resolutions; ~~the lifted~~ two such lifts are homotopic.

Proof :-

① First prove some facts about abelian groups.

• An injective abelian group \equiv A divisible abelian group.

• If A is an abelian group then we can embed A into an injective abelian group :- Let $A^\vee = \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$.

Let F be a free group such that $F^\bullet \rightarrow A^\vee \rightarrow 0$.

Dualize to get $A^{\vee\vee} \hookrightarrow F^\vee$. Now $A \hookrightarrow A^{\vee\vee}$

$\therefore A \hookrightarrow F^\vee$ and dual of free is injective.

Now, given an R -module M , let M^+ be the underlying abelian group.

Let I be an injective abelian group such that $M^+ \hookrightarrow I$.

Then

$$M \cong \text{Hom}_R(R, M) \subseteq \text{Hom}_{\mathbb{Z}}(R, M^+) \subseteq \text{Hom}_{\mathbb{Z}}(R, I) \quad (1)$$

check that $\text{Hom}_{\mathbb{Z}}(R, I)$ is an injective R -module.

Once every module can be embedded into an injective module, we get an injective resolution as :-

$$0 \rightarrow M \hookrightarrow I^0 \rightarrow I^1 \rightarrow \dots \rightarrow I^n \rightarrow I^{n+1}$$

$\searrow \quad \nearrow$ c^0 $\searrow \quad \nearrow$ c^n

where $c^0 = \text{coker}(M \hookrightarrow I^0)$, $c^n = \text{coker}(I^{n-1} \rightarrow I^n) \dots \text{etc} \dots$

Proof of ② is exactly like the corresponding statement for projective resolutions. Just reverse all arrows!

You will be repeatedly using the fact :-

$$I \text{ is injective} \iff \begin{array}{ccc} 0 & \rightarrow & A \hookrightarrow B \\ & & \downarrow \\ & & I \end{array} \leftarrow \text{dashed arrow}$$