Lecture-21
(Apr 2)

Cohomology of Spheres.

(Appendix on cohomology of the unit interval.)

Theorem

Let S^n = n-sphere = $\{(x_0, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$.

Assume $n \geq 1$.

$$H^q(S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0 \text{ or } n \\ 0, & q \notin \{0, n\}. \end{cases}$$

Proof:- The proof is by induction on n. But before we start the proof, we need the cohomology groups of S^0.

$S^0 = \{x_0 \in \mathbb{R} \mid x_0^2 = 1\} = \{\pm 1\}$. — disjoint union of two points.

A couple of easy observations:-

- If $X = X_1 \sqcup X_2$ then $H^q(X, A) = H^q(X_1, A) \oplus H^q(X_2, A)$
- If $X = \{*\}$ a singleton then $H^q(\{*\}, A) = \begin{cases} A, & q = 0 \\ 0, & q \geq 1. \end{cases}$

Hence we can conclude that

$$H^q(S^0, \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & q = 0 \\ 0, & q \geq 1. \end{cases}$$

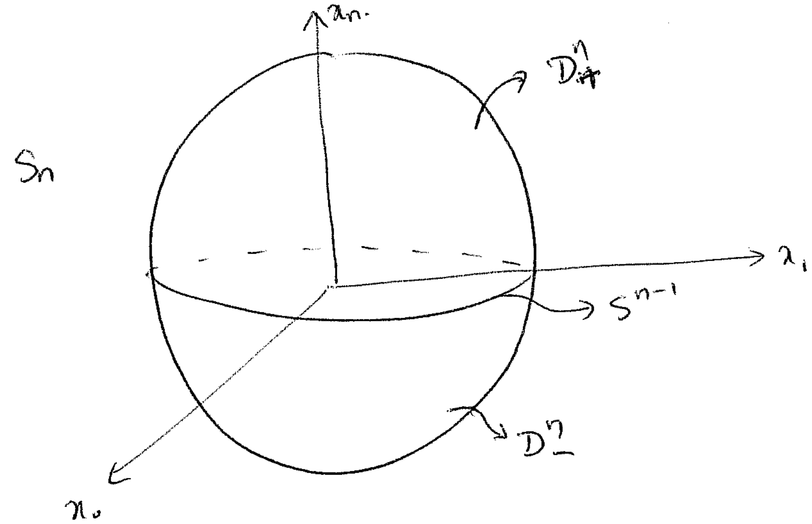
To set up the induction on n, we realize each S^n as a "northern hemisphere" & a "southern hemisphere", and the intersection of these two hemispheres is the "equator" which is in fact S^{n-1}.

More precisely, let

$$D^n_+ = \{(x_0, \dots, x_n) \in S^n \mid x_n \geq 0\}.$$

$$D^n_- = \{(x_0, \dots, x_n) \in S^n \mid x_n \leq 0\}$$

$$\partial D^n_+ = \partial D^n_- = D^n_+ \cap D^n_- = \{(x_0, \dots, x_{n-1}, 0) \in S^n\} \cong S^{n-1}.$$



- Consider the following sheaves on S^n :-

$\mathbb{Z}_{S^n}$ = constant sheaf on S^n determined by $\mathbb{Z}$.

$\mathbb{Z}_+ = i_{+*} \mathbb{Z}_{D^n_+}$ = direct image, via the inclusion $i_+ : D^n_+ \hookrightarrow S^n$ of the constant sheaf $\mathbb{Z}_{D^n_+}$ on D^n_+.

$\mathbb{Z}_- = i_{-*} \mathbb{Z}_{D^n_-}$ = - - - - - $i_- : D^n_- \hookrightarrow S^n$, - - - -

$\mathbb{Z}_{S^{n-1}} = i_* \mathbb{Z}_{S^{n-1}}$ = direct image, via $i : S^{n-1} \hookrightarrow S^n$, of the constant sheaf $\mathbb{Z}_{S^{n-1}}$ on S^{n-1}.

- Now consider the exact sequence of sheaves on S^n :-

$$\boxed{0 \longrightarrow \mathbb{Z}_{S^n} \xrightarrow{\alpha} \mathbb{Z}_+ \oplus \mathbb{Z}_- \xrightarrow{\beta} \mathbb{Z}_{S^{n-1}} \longrightarrow 0.} \qquad \text{——} (1).$$

The morphism α & β are:- $\quad U \underset{\text{open}}{\subset} S^n$

$$\alpha(U) : \mathbb{Z}_{S^n}(U) \longrightarrow \mathbb{Z}_+(U) \oplus \mathbb{Z}_-(U).$$
$$f \longmapsto (f|_{U\cap D^n_+},\ f|_{U\cap D^n_-})$$

where f is a $\mathbb{Z}$-valued locally constant fn. on S^n.

$$\beta(U) : \mathbb{Z}_+(U) \oplus \mathbb{Z}_-(U) \longrightarrow \mathbb{Z}_{S^{n-1}}(U).$$
$$(g, h) \longmapsto g|_{U\cap S^{n-1}} - h|_{U\cap S^{n-1}}$$

g - loc. constant $\mathbb{Z}$-valued fn. on $D^n_+ \cap U$.

h - " " " " $D^n_- \cap U$.

We need to check that the sequence (1) is exact. One can check exactness by checking the corresponding sequence of stalks is exact for all points of S^n. We will need the following detail:-

(exercise!) If $A \overset{i}{\hookrightarrow} X$ is a closed embedding & $\mathcal{F}$ is a sheaf on A then $(i_*\mathcal{F})_x = \begin{cases} \mathcal{F}_x, & x \in A \\ 0, & x \notin A \end{cases}$

In particular:-

$$\mathbb{Z}_{+,x} = \begin{cases} \mathbb{Z}, & x \in D^n_+ \\ 0, & x \notin D^n_+ \end{cases} \qquad \mathbb{Z}_{S^{n-1},x} = \begin{cases} \mathbb{Z}, & x \in S^{n-1} \\ 0, & x \notin S^{n-1}. \end{cases}$$

$$\mathbb{Z}_{-,x} = \begin{cases} \mathbb{Z}, & x \in D^n_- \\ 0, & x \notin D^n_- \end{cases}$$

$$(\mathbb{Z}_+ \oplus \mathbb{Z}_-)_x = \begin{cases} \mathbb{Z}, & x \in \mathrm{Int}(D^n_+) \\ \mathbb{Z}\oplus\mathbb{Z}, & x \in S^{n-1} \\ \mathbb{Z}, & x \in \mathrm{Int}(D^n_-) \end{cases}$$

The sequence of stalks for (1) looks like:-

$x \in \mathrm{Int}(D^n_+)$ $\quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus 0 \longrightarrow 0 \longrightarrow 0$

$\qquad\qquad n \longmapsto (n, 0)$

$x \in \mathrm{Int}(D^n_-)$ $\quad 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \oplus \mathbb{Z} \longrightarrow 0 \longrightarrow 0$ $\qquad$ Exact!

$\qquad\qquad n \longmapsto (0, n)$

$x \in S^{n-1}$ $\quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$

$\qquad\qquad n \longmapsto (n, n)$

$\qquad\qquad (n, m) \longmapsto n - m.$

- Now consider the long exact sequence in cohomology corresponding to the short exact sequence (1) :-

$$0 \to H^0(S^n, \mathbb{Z}_{S^n}) \to H^0(S^n, \mathbb{Z}^n_+ \oplus \mathbb{Z}^n_-) \to H^0(S^n, \mathbb{Z}_{S^{n-1}}) \to H^1(S^n, \mathbb{Z}_{S^n}) \to H^1(S^n, \mathbb{Z}_+ \oplus \mathbb{Z}_-) \to \cdots$$

and more generally,

$$\cdots \to H^{q-1}(S^n, \mathbb{Z}_{S^{n-1}}) \to H^q(S^n, \mathbb{Z}_{S^n}) \to H^q(S^n, \mathbb{Z}_+ \oplus \mathbb{Z}_-) \to H^q(S^n, \mathbb{Z}_{S^{n-1}}) \to H^{q+1}(S^n, \mathbb{Z}_{S^n}) \to \cdots$$

Make the following observations:-

- $\boxed{H^q(X, \mathcal{F} \oplus \mathcal{G}) = H^q(X, \mathcal{F}) \oplus H^q(X, \mathcal{G}).}$ — Derived functors of an additive functor are additive.

- $\boxed{\text{If } A \overset{i}{\hookrightarrow} X \text{ is a closed embedding then } H^q(X, i_*\mathcal{F}) = H^q(A, \mathcal{F}).}$

(Since $i : A \hookrightarrow X$ is a closed embedding, $i_* : S_A \to S_X$ is an exact functor $\Rightarrow R^q i_* \mathcal{F} = 0 \ \forall q \geq 1$

$\Rightarrow H^q(X, i_*\mathcal{F}) \longrightarrow H^q(A, \mathcal{F})$ is an isomorphism.)

- $\boxed{\text{If } X \text{ is connected then } H^0(X, \mathbb{Z}) = \mathbb{Z}.}$

• Also, use the homotopy axiom for D^n_+, D^n_- & conclude:-

$$\begin{aligned} H^q(S^n, \mathbb{Z}_+ \oplus \mathbb{Z}_-) &= H^q(S^n, \mathbb{Z}_+) \oplus H^q(S^n, \mathbb{Z}_-) \\ &= H^q(D^n_+, \mathbb{Z}) \oplus H^q(D^n_-, \mathbb{Z}) \\ &= \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & q = 0 \\ 0, & q \geq 1. \end{cases} \end{aligned}$$

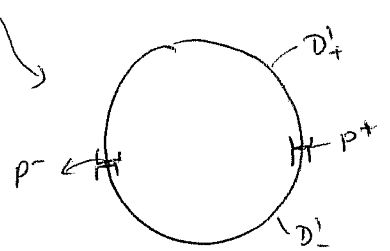
Putting all these observations, (2) & (3) may be rewritten as:-

$(n \geq 1)$

$$(2') \qquad 0 \to H^0(S^n, \mathbb{Z}) \to H^0(S^n, \mathbb{Z}^+ \oplus \mathbb{Z}^-) \to H^0(S^{n-1}, \mathbb{Z}) \to H^1(S^n, \mathbb{Z}) \to \underset{\substack{\| \\ 0.}}{H^1(S^n, \mathbb{Z}^+ \oplus \mathbb{Z}^-)}$$

For $n = 1$ we get

$$\begin{array}{ccccccccc} 0 \to & H^0(S^1, \mathbb{Z}) & \to & \underbrace{H^0(S^1, \mathbb{Z}^+ \oplus \mathbb{Z}^-)} & \to & \underbrace{H^0(S^0, \mathbb{Z})} & \to & H^1(S^1, \mathbb{Z}) & \to 0. \\ & \| & & \| & & \| & & \| & \\ 0 \to & \mathbb{Z} & \to & \mathbb{Z} \oplus \mathbb{Z} & \to & \mathbb{Z} \oplus \mathbb{Z} & \to & ? & \to 0. \\ & n & \mapsto & (n, n). & & & & & \\ & & & (n, m) & \mapsto & (n-m, n-m) & & & \end{array}$$



$S^0 = \{p^-, p^+\}$.

Conclusion:- $H^1(S^1, \mathbb{Z}) = \dfrac{\mathbb{Z} \oplus \mathbb{Z}}{\Delta \mathbb{Z}} \simeq \mathbb{Z}$.

$$\boxed{H^1(S^1, \mathbb{Z}) = \mathbb{Z}.}$$

For $n \geq 2$ we get

$$\begin{array}{ccccccccc} 0 \to & H^0(S^n, \mathbb{Z}) & \to & H^0(S^n, \mathbb{Z}^+ \oplus \mathbb{Z}^-) & \to & H^0(S^{n-1}, \mathbb{Z}) & \to & H^1(S^n, \mathbb{Z}) & \to 0 \\ & \| & & \| & & \| & & \| & \\ & \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} & & ? & \end{array}$$

Hence $\boxed{H^1(S^n, \mathbb{Z}) = 0 \quad \forall n \geq 2.}$

Now we look at (3'):-

$$\to H^{q-1}(S^{n-1}, \mathbb{Z}) \to H^q(S^n, \mathbb{Z}) \to H^q(S^n, \mathbb{Z}_+ \oplus \mathbb{Z}_-) \to H^q(S^{n-1}, \mathbb{Z}) \to H^{q+1}(S^n, \mathbb{Z}) \to H^{q+1}(S^n, \mathbb{Z}_+ \oplus \mathbb{Z}_-) \to .$$

If $q \geq 1$ then $H^q(S^n, \mathbb{Z}_+ \oplus \mathbb{Z}_-) = H^{q+1}(S^n, \mathbb{Z}_+ \oplus \mathbb{Z}_-) = 0$. (By homotopy axiom)

$$\Rightarrow \boxed{H^{q+1}(S^n, \mathbb{Z}) \cong H^q(S^{n-1}, \mathbb{Z}).} \quad \forall q \geq 1. \ (\forall n \geq 1)$$

In particular:-

$$\boxed{H^n(S^n, \mathbb{Z}) \approx H^{n-1}(S^{n-1}, \mathbb{Z}) \approx \cdots \approx H^1(S^1, \mathbb{Z}) \approx \mathbb{Z}.}$$

If $1 \leq q < n$ then

$$H^q(S^n, \mathbb{Z}) \approx H^{q-1}(S^{n-1}, \mathbb{Z}) \approx \cdots \approx H^1(S^{n-q+1}, \mathbb{Z}) \approx 0.$$

since $n-q+1 \geq 2$.

$$\Rightarrow \boxed{H^q(S^n, \mathbb{Z}) = 0 \quad \text{if } 1 \leq q < n.}$$

Further, $H^{q+1}(S^1, \mathbb{Z}) = H^q(S^0, \mathbb{Z}) = 0$ if $q \geq 1$.

$$\Rightarrow \boxed{H^q(S^1, \mathbb{Z}) = 0 \quad \text{if } q \geq 2.}$$

So, if $q > n$ then

$$H^q(S^n, \mathbb{Z}) \approx \cdots \approx H^{q-n+1}(S^1, \mathbb{Z}) \approx 0.$$

$$\Rightarrow \boxed{H^q(S^n, \mathbb{Z}) = 0 \quad \text{if } q > n.}$$

This concludes the proof!

Appendix:- Cohomology of the unit interval.

Theorem:-

$$\boxed{I = [0,1], \qquad H^q(I, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0 \\ 0, & q \geq 1. \end{cases}}$$

(This is needed in the proof of the homotopy axiom.)

Let us say that a sheaf $\mathcal{A}$ on I satisfies $(\mathcal{E})$ if for all open intervals $J \subset I$ the map $\mathcal{A}(I) \longrightarrow \mathcal{A}(J)$ is surjective.

Note:- This does not mean $\mathcal{A}$ is flabby. This condition is weaker than flabby.

Examples:-

- If $\mathcal{A} = A_I$ = constant sheaf determined by an abelian group then $\mathcal{A}$ satisfies $(\mathcal{E})$.

$$\begin{array}{ccc} A_I(I) & \longrightarrow\!\!\!\!\!\rightarrow & A_I(J) \\ \| & & \| \\ \{f : [0,1] \longrightarrow A\} & & \{f : J \longrightarrow A\} \end{array}$$

J is connected!

- If $\mathcal{A} = \mathcal{I}$ an injective sheaf then $\mathcal{A}$ satisfies $(\mathcal{E})$

Injective $\Rightarrow$ Flabby $\Rightarrow (\mathcal{E})$.

Lemma 1-

If $\mathcal{A}$ satisfies $(\mathcal{E})$ and $0 \to \mathcal{A} \to \mathcal{F} \to \mathcal{G} \to 0$ is an exact sequence of sheaves on I then $\mathcal{F}(J) \longrightarrow \mathcal{G}(J)$ is surjective. for any open interval $J \subset I$.

Pf:- Let $s \in \mathcal{G}(J)$. Since $\mathcal{F} \twoheadrightarrow \mathcal{G}$, s can be locally lifted.

Let $\Sigma = \{ (U, t) \mid U \underset{\text{open}}{\subset} J,\ U\text{-interval},\ t \in \mathcal{F}(U),\ t \mapsto s|_U \}$.

Put a partial order $\leq$ on Σ :- $(U_1, t_1) \leq (U_2, t_2)$ if $U_1 \subset U_2$, $t_2|_{U_1} = t_1$

Note that Zorn's lemma is applicable and let (U_0, t_0) be a maximal element of Σ.

If ~~…~~ $U_0 = J$ then we are done. If $U_0 \subsetneq J$ then let $x \in \overline{U_0} \setminus U_0$ (and in fact $x \in J$). Take a small interval V, $x \in V \subset J$

and let $t_1 \in \mathcal{F}(V)$ s.t. $t_1 \longmapsto s|_V$.

On $U_0 \cap V$, $t_0|_{U_0\cap V} - t_1|_{U_0\cap V} \longmapsto s|_{U_0\cap V} - s|_{U_0\cap V} = 0$ in $\mathcal{G}(U_0\cap V)$

$\Rightarrow \exists\, r \in \mathcal{A}(U_0\cap V)$ s.t. $r \longmapsto t_0|_{U_0\cap V} - t_1|_{U_0\cap V}$ in $\mathcal{F}(U_0\cap V)$

Since $\mathcal{A}$ satisfies $(\mathcal{E})$, $\exists\, r_1' \in \mathcal{A}(I)$ s.t. ~~…~~ $r_1'|_{U_0\cap V} = r$.

Let $r_1 = r_1'|_{\mathcal{A}(V)}$.

Let $t_1' \in \mathcal{F}(V)$ ~~…~~ $t_1' = t_1 + \alpha(r_1)$ (if $\alpha : \mathcal{A} \to \mathcal{F}$)

Then $t_0|_{U_0\cap V} - t_1'|_{U_0\cap V} = t_0|_{U_0\cap V} - t_1|_{U_0\cap V} - \alpha(r_1)|_{U_0\cap V} = 0$.

$\Rightarrow$ $t_0 \in \mathcal{F}(U_0)$ & $t_1' \in \mathcal{F}(V_1)$ patch up

$\Rightarrow$ $\exists\, t \in \mathcal{F}(U_0 \cup V_1)$ s.t. $t \longmapsto s|_{U_0\cup V_1}$, $U_0 \subsetneq U_0 \cup V_1$ -- contradict maximality.

Lemma 2

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be a short exact sequence of sheaves on I.

If $\mathcal{A}$ & $\mathcal{B}$ satisfy $(\mathcal{E})$ then so does $\mathcal{C}$.

Pf:- Let J be any open interval in I, <u>want</u>:- $\mathcal{C}(I) \twoheadrightarrow \mathcal{C}(J)$.

Consider the diagram:-

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{A}(I) & \to & \mathcal{B}(I) & \to & \mathcal{C}(I) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{A}(J) & \to & \mathcal{B}(J) & \to & \mathcal{C}(J) & \to & 0
\end{array}$$

- Rows are exact by Lemma 1, since $\mathcal{A}$ satisfies $(\mathcal{E})$
- Vertical arrows are restriction maps.
- $\mathcal{B}(I) \twoheadrightarrow \mathcal{B}(J)$ since $\mathcal{B}$ satisfies $(\mathcal{E})$ $\Longrightarrow$ $\mathcal{C}(I) \twoheadrightarrow \mathcal{C}(J)$.

Lemma 3

$$\boxed{\text{Let } \mathcal{A} \text{ be any sheaf on } I \text{ that satisfies } (\mathcal{E}) \text{ then } \mathcal{A} \text{ is acyclic, i.e. } H^q(I, \mathcal{A}) = 0 \quad \forall\, q \geq 1}$$

Pf:- Embed $\mathcal{A}$ into an injective sheaf $\mathcal{J}$ and let $\mathcal{B} = \mathcal{J}/\mathcal{A}$:-

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{J} \longrightarrow \mathcal{B} \longrightarrow 0.$$

$\mathcal{J}$ - injective $\Rightarrow$ $\mathcal{J}$ - flabby $\Rightarrow$ $\mathcal{J}$ - satisfies $(\mathcal{E})$

By Lemma 2, since $\mathcal{A}$ & $\mathcal{J}$ satisfy $(\mathcal{E})$ $\Longrightarrow$ $\mathcal{B}$ satisfies $(\mathcal{E})$.

<u>Proof is by induction on</u> q :-

$q = 1$, $\quad 0 \longrightarrow \mathcal{A}(I) \longrightarrow \mathcal{J}(I) \longrightarrow \mathcal{B}(I) \longrightarrow 0$ is exact since $\mathcal{A}$ satisfies $(\mathcal{E})$

The long exact sequence in cohomology looks like:-

$$0 \to \mathcal{A}(I) \longrightarrow \mathcal{J}(I) \twoheadrightarrow \mathcal{B}(I) \longrightarrow H^1(I, \mathcal{A}) \longrightarrow \underset{\substack{\| \\ 0}}{H^1(I, \mathcal{J})} \longrightarrow \cdots$$

$\left.\begin{array}{l}\mathcal{J}\text{ - injective} \Rightarrow \mathcal{J}\text{ - acyclic} \Rightarrow H^1(I, \mathcal{J}) = 0 \\ \mathcal{J}(I) \to \mathcal{B}(I) \text{ - surjective}\end{array}\right\} \Rightarrow H^1(I, \mathcal{A}) = 0$

Assume that we have proved Lemma 3 for $q = 1$ & all sheaves $\mathcal{A}$ that satisfy $(\mathcal{E})$.

$q \geq 2$:-

$$\longrightarrow \underset{\substack{\| \\ 0}}{H^1(I, \mathcal{J})} \longrightarrow H^1(I, \mathcal{B}) \longrightarrow H^2(I, \mathcal{A}) \longrightarrow \underset{\substack{\| \\ 0}}{H^2(I, \mathcal{J})} \longrightarrow \cdots$$

$\Rightarrow \quad H^2(I, \mathcal{A}) = H^1(I, \mathcal{B}) = 0$ since $\mathcal{B}$ also satisfies $(\mathcal{E})$.

More generally $\quad H^q(I, \mathcal{A}) = H^{q-1}(I, \mathcal{B}) = 0 \quad (q \geq 2)$

Finally, since any constant sheaf A_I satisfies $(\mathcal{E})$ we deduce that

$$H^q(I, A_I) = 0 \qquad \forall\, q \geq 1.$$

and in particular,

$$H^q(I, \mathbb{Z}) = 0 \qquad \forall\, q \geq 1.$$

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