## Homotopy axiom

- If a space $X$ can be contineously deformed into another space $Y$ then $X$ + $Y$ have the same cohomology groups

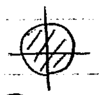
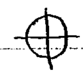
- If $f, g : X \to Y$, cont. maps, $f$ is "homotopic" to $g$, then $H^q(f), H^q(g) : H^q(Y, A) \to H^q(X, A)$ are equal as homormorphisms $( H^q(f) = H^q(g) )$

**Def:** $X, Y$ : top spaces, $I = [0, 1]$ : unit interval
$f, g : X \to Y$ are cont. maps
We say $f$ is homotopic to $g$ if $\exists$ a map $F : X \times I \to Y$ s.t.
$F|_{X \times \{0\}} = f$, $F|_{X \times \{1\}} = g$ (or $F(x, 0) = f(x)$, $F(x, 1) = g(x)$ $\forall x \in X$)

$$
\begin{tikzcd}
X \times I \arrow[dr, dashed, "\exists F"] & X \arrow[l, hook', "(x,1) \mapsfrom x"'] \arrow[d, "g"] \\
X \arrow[u, hook, "(x,0) \mapsfrom x"] \arrow[r, "f"'] & Y
\end{tikzcd}
$$

Notation : $f \simeq g$

**Def:** $X$ is said to be contractible if $1_X \simeq$ (constant map)

- $\mathbb{R}^n$ : Euclidean $n$-space
$D^n := \{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum x_i^2 \le 1 \}$ · $D^1 = [-1, 1]$, $D^2 =$ 
$S^n := \{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1 \}$ $S^1 =$ , $S^2 =$ 

NOTE : $\partial D^{n+1} = S^n$

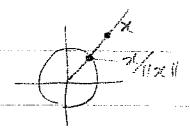
**Example:** · $\mathbb{R}^n$ is contractible, $F : \mathbb{R}^n \times I \to \mathbb{R}^n$, $F(\vec{x}, t) = t\vec{x}$
· $D^n$ is contractible, same homotopy

**Def:** • $f : X \to Y$ is said to be a <u>homotopy equivalence</u> if $\exists g : Y \to X$ s.t. $f \circ g \simeq 1_Y$ + $g \circ f \simeq 1_X$.

• $X, Y$ are said to be <u>homotopic</u> if $\exists$ homotopy equivalence $f : X \to Y$.

**Example:** $\mathbb{R}^{n+1} - \{0\}$ is homotopic to $S^n$.

$S^n \xhookrightarrow{i} \mathbb{R}^{n+1} \setminus \{0\}$ is a homotopy equivalence.
(Take $g : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ as $g(\vec{x}) = \vec{x}/\|\vec{x}\|$)




**Theorem 1** (Homotopy axiom)
$f, g : X \to Y$ : homotopic maps, $A$ : abelian group
Then $H^q(f) = H^q(g) : H^q(Y, A) \to H^q(X, A)$.

**Cor:** $X, Y$ : homotopic spaces. Then $H^q(X, A) \simeq H^q(Y, A)$.

*Proof*
$X, Y$ : homotopic $\Rightarrow \exists f : X \to Y, \exists g : Y \to X$ s.t. $f \circ g \simeq 1_Y$, $g \circ f \simeq 1_X$.

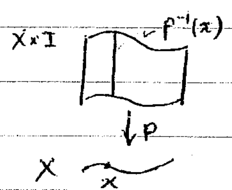
$\Rightarrow H^q(f) : H^q(Y, A) \to H^q(X, A)$ is an isomorphism

$$H^q(g) \circ H^q(f) \underset{\uparrow}{=} H^q(1_Y) = 1_{H^q(Y, A)}$$
by Homotopy axiom

**Theorem 2:** $X$ : any top. sp. $P : X \times I \to X$ given by $P(x, t) = x$.
Then $P$ induces an isomorphism in cohomology
i.e. $H^q(P) : H^q(X, A) \to H^q(X \times I, A)$ is an isomorphism $\forall q$.

*Proof*
<u>Step 1</u>: The conclusion of the proper base change theorem holds for the map $P$. i.e. for any sheaf $\mathcal{F}$ on $X \times I$ + $x \in X$, $R^q P_*(\mathcal{F})_x = H^q(P^{-1}(x), \mathcal{F}|_{P^{-1}(x)})$

For a proof of this statement look up Harder's book. ← (Assume this.)




Note $\forall x \in X$, $P^{-1}(x) = \{x\} \times I \simeq I$

- If $\mathcal{F} = A_{X\times I}$, constant sheaf determined by $A$, then $\mathcal{F}|_{p^{-1}(x)} = A_{p^{-1}(x)}$.

Then $H^q(p^{-1}(x), A) = H^q(I, A) = 0 \quad \forall q \geq 1$ ( exercise )
$\Rightarrow R^q p_*(A_{X\times I}) = 0 \quad \forall q \geq 1$.

<u>Step 2</u>: Recall $f: X \to Y$, $R^q f_*(\mathcal{F}) = 0 \quad \forall q \geq 1$
$\Rightarrow H^q(Y, f_*\mathcal{F}) \simeq H^q(X, \mathcal{F})$.

Apply this to $p: X \times I \to X$, $\mathcal{F} = A_{X\times I}$,
we get $H^q(X, p_* A_{X\times I}) \simeq H^q(X \times I, A)$

(NOTE) : $p_*(A_{X\times I}) = A_X$

$\Rightarrow H^q(X, A) \simeq H^q(X \times I, A)$ //

Cor: For any $t \in I$, the inclusion $i_t : X \longrightarrow X \times I$ given by $i_t(x) = (x, t)$ induces an isomorphism in cohomology.

Proof

$$\begin{tikzcd} X \arrow[r, hook, "i_t"] \arrow[dr, "1_X"'] & X \times I \arrow[d, "p"] \\ & X \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} H^q(X, A) & H^q(X \times I, A) \arrow[l, "H^q(i_t)"'] \\ & H^q(X, A) \arrow[ul, "\simeq", "H^q(1_X)"'] \arrow[u, "H^q(p)"', "S"] \end{tikzcd}$$

by Thm.

$\therefore H^q(i_t)$ is an isomorphism //

Proof of homotopy axiom

$f, g : X \to Y$ : homotopic maps
Let $F : X \times I \to Y$ be the homotopy from $f$ to $g$.
Consider

$$\begin{tikzcd} X \arrow[r, hook, "i_0"] \arrow[dr, "f"'] & X \times I \arrow[d, "F"] \\ & Y \end{tikzcd} \quad + \quad \begin{tikzcd} X \arrow[r, hook, "i_1"] \arrow[dr, "g"'] & X \times I \arrow[d, "F"] \\ & Y \end{tikzcd}$$

$x \mapsto (x, 0)$

$\Rightarrow H^q(f) = H^q(i_0) \circ H^q(F)$ , $H^q(g) = H^q(i_1) \circ H^q(F)$.