

## Lecture-2      Projective & Inductive Limits.

Projective limit = Inverse limit

( ) Inductive limit = Direct limit

### Defn Projective System / Inverse system

Considers an ordered set  $I = (I, \leq)$ .

This is just a partially ordered set.

Let  $\mathcal{C}$  be any category.

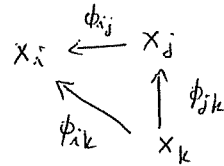
A projective/inverse system is  $(\{X_i\}_{i \in I}, \{\phi_{ij}\})$  consisting of

•  $X_i \in \text{Ob}(\mathcal{C}) \quad \forall i \in I$

•  $\forall i, j \in I$ , with  $i \leq j$ , we are given  $\phi_{ij} \in \text{Hom}_{\mathcal{C}}(X_j, X_i)$  such that

•  $\phi_{ii} = 1_{X_i}$

•  $i \leq j \leq k \Rightarrow$



$$\phi_{ij} \circ \phi_{jk} = \phi_{ik}$$

( ) Note:- The map  $\phi_{ij} : X_j \rightarrow X_i$  is in the "reverse" direction.

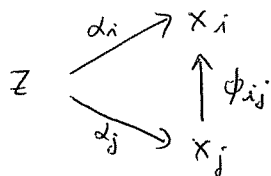
• A projective system is a contravariant functor  $I \rightarrow \mathcal{C}$ .

~~Defn~~

Given a projective system  $(\{X_i\}, \phi_{ij})$  and given any  $Z \in \text{Ob}(\mathcal{C})$

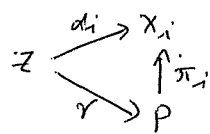
we can define  $\alpha \in \text{Hom}_{\mathcal{C}}(Z, (\{X_i\}, \phi_{ij}))$  as a

family  $\alpha = \{\alpha_i\}$ ,  $\alpha_i : Z \rightarrow X_i$ , such that

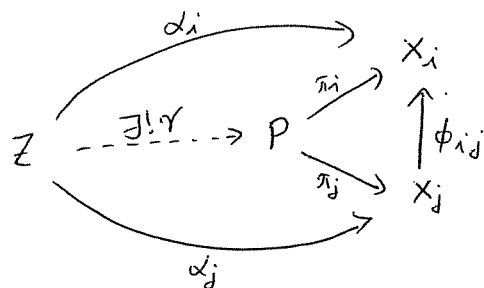


$\forall i \leq j$ .

Definition 1 The projective limit  $(\{X_i\}, \phi_{ij})$ , denoted  $\varprojlim X_i$ , is an object  $P$ , and  $\pi \in \text{Hom}_{\mathcal{C}}(P, (\{X_i\}, \phi_{ij}))$  such that ~~any~~ any  $\alpha \in \text{Hom}_{\mathcal{C}}(Z, (\{X_i\}, \phi_{ij}))$  uniquely factors through  $P$  and  $\pi$ ; meaning that,  $\exists! \gamma \rightarrow P$  such that



This definition is very conveniently expressed as ~~that~~ the ~~existence~~ existence of a unique dotted arrow in



Definition 1' An object  $P$ , together with an element  $\pi \in \text{Hom}_{\mathcal{C}}(P, (\{X_i\}, \phi_{ij}))$ , is called a projective limit of  $(\{X_i\}, \phi_{ij})$  if for any  $Z \in \text{ob}(\mathcal{C})$

the map

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(Z, P) & \xrightarrow{\quad} & \text{Hom}_{\mathcal{C}}(Z, (\{X_i\}, \phi_{ij})) \\ \psi \downarrow & \xrightarrow{\quad} & \{\pi_i \circ \psi\}_{i \in I} \end{array}$$

is a bijection.

Existence of dotted arrow is the surjection part.

Uniqueness " " " injection part.

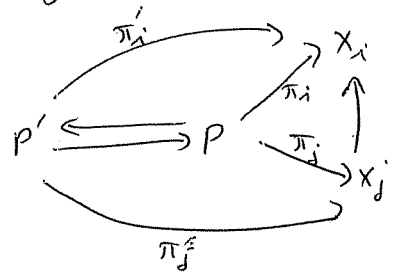
Note:- Definition 1' can be rephrased as that the functor

$$\text{Hom}_{\mathcal{C}}(\_, (\{X_i\}, \phi_{ij})) : \mathcal{C} \longrightarrow \text{SETS}$$

is represented by  $(P, \pi)$ .

Yoneda's Lemma :- If  $(P, \pi)$  exists then it is unique up to a canonical isomorphism.

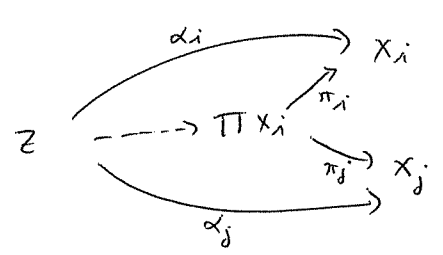
This means that if  $(P', \pi')$  is another candidate for the projective limit of  $(\{X_i\}, \phi_{ij})$  then  $\exists!$  isomorphism  $P \rightarrow P'$ . You can see this by staring at the diagrams:-



Examples 1:-

① Projective limit generalizes the notion of product of sets:-

Let  $\mathcal{C} = \text{SETS}$ .  
 Let  $I$  be an indexing set with the trivial order :-  $i \leq j \Leftrightarrow i = j$ .  
 Consider a projective system in  $\mathcal{C}$  based on  $I$  :- this is just a collection of sets  $\{X_i\}$ .  
 The projective limit is  $P = \prod X_i$ , with  $\pi = (\pi_i)$  and  $\pi_i$  is projecting to the  $i^{\text{th}}$  coordinate. Given any  $Z$ ,  $\phi_i : Z \rightarrow X_i$



The unique dotted arrow must be  $Z \rightarrow (\alpha_i(Z)) \quad \forall Z \in \mathcal{Z}$ .

② Products may or may not exist in very familiar looking categories:-

- GROUPS :- Products exist.
- RINGS :- " " .
- FIELDS :- " do not exist.

③ Projective completion of  $\mathbb{Z}$  :-

Conventions:-

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

$$\mathbb{N}_+ = \{1, 2, 3, \dots\}$$

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}$$

Ordered set:-  $\mathbb{N}_+$  with  
 $n \leq m$  iff  $n|m$ .

Projective system:-  $n \leq m$ , i.e.,  $n|m$ ,  $m\mathbb{Z} \subset n\mathbb{Z}$   
 $\therefore$  There is a canonical map  $\mathbb{Z}/m\mathbb{Z} \xrightarrow{\phi_{mn}} \mathbb{Z}/n\mathbb{Z}$  (induced from identity on  $\mathbb{Z}$ )  
 $(\{\mathbb{Z}/n\mathbb{Z}\}, \phi_{nm})$ .

Projective limit:-  $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$  the projection map  $\phi_n: \hat{\mathbb{Z}} \rightarrow \mathbb{Z}/n\mathbb{Z}$ .

(All this may be viewed in the category of commutative rings).

concretely, we may construct  $\hat{\mathbb{Z}}$  as:-

$$\hat{\mathbb{Z}} = \{ (\dots, x_n, \dots)_{n \in \mathbb{N}_+} \mid \text{~~when } n \leq m \text{ then } x_m \equiv x_n \pmod{n} \text{ if } n|m \}~~$$

and  $\phi_n: \hat{\mathbb{Z}} \rightarrow \mathbb{Z}/n$  is simply projecting to the  $n^{\text{th}}$ -coordinate.

④  $p$ -adic integers: For a prime  $p$ .

Ordered set:-  $\mathbb{N}_+$  with usual order.  $\leq$ .

Projective system:-  $\mathbb{Z}/p^n\mathbb{Z}$ ,  $n \leq m$  then  $\mathbb{Z}/p^m\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$  is the canonical map.

$$\mathbb{Z}_p := \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$$

concretely,  $\mathbb{Z}_p = \{ (\dots, a_n, \dots) : a_m \equiv a_n \pmod{p^n} \text{ if } n \leq m \}$ .

An exercise with Chinese remainders theorem says that

$$\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$$

## § Direct Limits

- Projective limit generalizes the usual set-theoretic product
- Direct limit " " notion of disjoint union.
- Direct limit is the "dual" notion of projective limit.  
Just reverse all arrows.

### Definition :-

•  $(I, \leq)$  be an ordered set,  $\mathcal{C}$  any category.

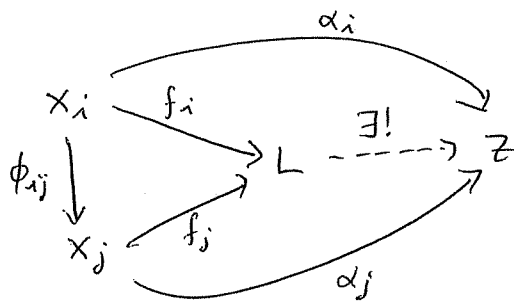
• An inductive system in  $\mathcal{C}$  based on  $I$  is  $(\{X_i\}, \psi_{ij})$  where

•  $X_i \in \text{Ob}(\mathcal{C}) \forall i,$

•  $\forall i \leq j$  we are given  $\psi_{ij} \in \text{Hom}_{\mathcal{C}}(X_i, X_j)$  (same direction)

So an inductive system is a covariant functor from  $I \rightarrow \mathcal{C}$ .

• The inductive limit or direct limit of the inductive system  $(\{X_i\}, \psi_{ij})$  consists of  $L = \varinjlim X_i$  and  $f_i: X_i \rightarrow L$  such that



To concretely describe the elements of  $L$ , it is helpful to have the ordered set to be a directed set.

Directed set is an ordered set  $(I, \leq)$  such that ~~given~~ for any  $i, j \in I$ , there exists  $k \in I$  such that  $i \leq k, j \leq k$ .

Elements of  $L$  :- Take the disjoint union  $\coprod_i X_i$   
modulo the equivalence relation :-

$$x_i \sim x_j \Leftrightarrow \exists k \ni i \leq k, j \leq k \text{ \& } \phi_{ik}(x_i) = \phi_{jk}(x_j).$$

Example:-

Let  $X$  be a topological space. Fix  $x_0 \in X$ .

Directed set :- All open subsets of  $X$  containing  $x_0$ , ordered by reverse inclusion.

$$U \leq V \Leftrightarrow U \supset V.$$

Direct/inductive system (of  $\mathbb{R}$ -vector spaces) :-

$\mathcal{C}(U)$  = all continuous  $\mathbb{R}$ -valued functions on  $U$ .

$U \leq V$ ,  $\phi_{UV} : \mathcal{C}(U) \rightarrow \mathcal{C}(V)$  is the restriction map.

Direct limit of this system exists and is denoted

$$\mathcal{C}_{x_0} = \varinjlim_{x_0 \in U} \mathcal{C}(U).$$

$\mathcal{C}_{x_0}$  is a  $\mathbb{R}$ -vector space.

$\mathcal{C}_{x_0}$  is called :- The stalk at  $x_0$  of the "sheaf" of continuous  $\mathbb{R}$ -valued functions on  $X$

OR

Germs of continuous functions at  $x_0$ .

Elements of  $\mathcal{C}_{x_0}$  are equivalence classes of pairs  $(U, f)$  where  $x_0 \in U \subset X$  and  $f: U \rightarrow \mathbb{R}$  is continuous and the equivalence relation is:-

$$(U, f) \sim (V, g) \Leftrightarrow f|_{U \cap V} = g|_{U \cap V}.$$