Lecture-2 Projective & Inductive Limits.

Projective limit $=$ Inverse limit

Inductive limit $=$ Direct limit

Defn Projective System / Inverse system

Consider an ordered set $\mathcal{I} = (I, \leq)$. This is just a partially ordered set.

Let $\mathcal{C}$ be any category.

A projective / inverse system is $(\{X_i\}_{i \in I}, \{\phi_{ij}\})$ consisting of

- $X_i \in Ob(\mathcal{C}) \quad \forall i \in I$
- $\forall i, j \in I$, with $i \leq j$, we are given $\phi_{ij} \in Hom_{\mathcal{C}}(X_j, X_i)$ such that
  - $\phi_{ii} = 1_{X_i}$
  - $i \leq j \leq k \implies$

$$\begin{tikzcd} X_i & X_j \arrow[l, "\phi_{ij}"'] \\ & X_k \arrow[u, "\phi_{jk}"'] \arrow[ul, "\phi_{ik}"] \end{tikzcd} \qquad \phi_{ij} \circ \phi_{jk} = \phi_{ik}$$

Note:- • The map $\phi_{ij} : X_j \longrightarrow X_i$ is in the "reverse" direction.

- A projective system is a contravariant functor. $\mathcal{I} \longrightarrow \mathcal{C}$.

Given a projective system $(\{X_i\}, \phi_{ij})$ and given any $Z \in Ob(\mathcal{C})$ we can define $\alpha \in Hom_{\mathcal{C}}(Z, (\{X_i\}, \phi_{ij}))$ as a family $\alpha = \{\alpha_i\}$, $\alpha_i : Z \longrightarrow X_i$, such that

$$\begin{tikzcd} & X_i \\ Z \arrow[ur, "\alpha_i"] \arrow[dr, "\alpha_j"'] & \\ & X_j \arrow[uu, "\phi_{ij}"'] \end{tikzcd} \qquad \forall i \leq j.$$

<u>Definition 1</u> The projective limit $(\{X_i\}, \phi_{ij})$, denoted $\varprojlim_i X_i$, is an object P, and $\pi \in \mathrm{Hom}_{\mathcal{C}}(P, (\{X_i\}, \phi_{ij}))$ such that any $\alpha \in \mathrm{Hom}_{\mathcal{C}}(Z, (\{X_i\}, \phi_{ij}))$ uniquely factors through P and π; meaning that, $\exists!\ Z \xrightarrow{\gamma} P$ such that

$$\begin{tikzcd} & X_i \\ Z \arrow[ur, "\alpha_i"] \arrow[dr, "\gamma"'] & \\ & P \arrow[uu, "\pi_i"'] \end{tikzcd}$$

This definition is very conveniently expressed as ~~that~~ the ~~dotted~~ existence of a unique dotted arrow in

$$\begin{tikzcd} & & X_i \\ Z \arrow[rru, bend left, "\alpha_i"] \arrow[rrd, bend right, "\alpha_j"'] \arrow[r, dashed, "\exists!\gamma"] & P \arrow[ru, "\pi_i"] \arrow[rd, "\pi_j"'] & \\ & & X_j \arrow[uu, "\phi_{ij}"'] \end{tikzcd}$$

<u>Definition 1'</u> An object P, together with an element $\pi \in \mathrm{Hom}_{\mathcal{C}}(P, (\{X_i\}, \phi_{ij}))$, is called a <u>projective limit</u> of $(\{X_i\}, \phi_{ij})$ if for any $Z \in \mathrm{ob}(\mathcal{C})$ the map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(Z, P) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(Z, (\{X_i\}, \phi_{ij})) \\ \psi &\longmapsto \{\pi_i \circ \psi\}_{i \in I} \end{aligned}$$

is a bijection.

Existence of dotted arrow is the surjection part.

Uniqueness " " " injection part.

<u>Note</u>:- Definition 1' can be rephrased as that the functor

$$\mathrm{Hom}_{\mathcal{C}}(-, (\{X_i\}, \phi_{ij})) : \mathcal{C} \longrightarrow \mathrm{SETS}$$

is <u>represented</u> by (P, π).

<u>Yoneda's Lemma</u> :- If (P, π) exists then it is unique up to a canonical isomorphism.

This means that if (P', π') is another candidate for the projective limit of $(\{X_i\}, \phi_{ij})$ then $\exists!$ isomorphism $P \longrightarrow P'$. You can see this by staring at the diagrams:-

$$\begin{tikzcd}
 & & X_i \\
P' \arrow[r, shift left] \arrow[rru, bend left, "\pi_i'"] \arrow[rrd, bend right, "\pi_j'"'] & P \arrow[l, shift left] \arrow[ru, "\pi_i"] \arrow[rd, "\pi_j"'] & \\
 & & X_j \arrow[uu]
\end{tikzcd}$$

Examples:-

① <u>Projective limit generalizes the notion of product of sets:-</u>

Let $\mathcal{C}$ = SETS.

Let I be an indexing set with the trivial order:- $i \leq j \Leftrightarrow i = j$.

consider a projective system in $\mathcal{C}$ based on I :- this is just a collection of sets $\{X_i\}$.

The projective limit is $P = \prod X_i$, with $\pi = (\pi_i)$ and π_i is projecting to the i^{th} coordinate. Given any Z, $\alpha_i : Z \longrightarrow X_i$

$$\begin{tikzcd}
 & & X_i \\
Z \arrow[r, dashed] \arrow[rru, bend left, "\alpha_i"] \arrow[rrd, bend right, "\alpha_j"'] & \prod X_i \arrow[ru, "\pi_i"] \arrow[rd, "\pi_j"'] & \\
 & & X_j
\end{tikzcd}$$

The unique dotted arrow <u>must</u> be

$z \longmapsto (\alpha_i(z)) \quad \forall z \in Z$.

② Products may or may not exist in very familiar looking categories:-

GROUPS :- Products exist.

RINGS :- " " .

FIELDS :- " do not exist.

Conventions:-

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$

$\mathbb{N}_+ = \{1, 2, 3, \dots\}$

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}$

③ Projective completion of $\mathbb{Z}$:-

- Ordered set :- $\mathbb{N}_+$ with

  $n \leq m$ iff $n \mid m$.

- Projective system:- $n \leq m$, i.e., $n \mid m$, $m\mathbb{Z} \subset n\mathbb{Z}$

  $\therefore$ There is a canonical map $\mathbb{Z}/m\mathbb{Z} \xrightarrow{\phi_{mn}} \mathbb{Z}/n\mathbb{Z}$. (induced from identity on $\mathbb{Z}$)

  $$(\{\mathbb{Z}/n\mathbb{Z}\}, \phi_{nm}).$$

- Projective limit:- $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ the projection map $\phi_n : \hat{\mathbb{Z}} \longrightarrow \mathbb{Z}/n\mathbb{Z}$.

(All this may be viewed in the category of commutative rings).

concretely, we may construct $\hat{\mathbb{Z}}$ as:-

$$\hat{\mathbb{Z}} = \{(\dots, x_n, \dots)_{n \in \mathbb{N}_+} \mid x_m \equiv x_n \pmod{n} \text{ if } n \mid m\}$$

and $\phi_n : \hat{\mathbb{Z}} \longrightarrow \mathbb{Z}/n$ is simply projecting to the n^{th}-coordinate.

④ p-adic integers: Fix a prime p.

Ordered set :- $\mathbb{N}_+$ with usual order. $\leq$.

Projective system :- $\mathbb{Z}/p^n\mathbb{Z}$, $n \leq m$ then $\mathbb{Z}/p^m\mathbb{Z} \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$ is the canonical map.

$$\mathbb{Z}_p := \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}.$$

concretely, $\mathbb{Z}_p = \{(\dots, a_n, \dots) : a_m \equiv a_n \pmod{p^n} \text{ if } n \leq m\}$.

An exercise with Chinese remainder theorem says that

$$\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p.$$

§ Direct Limits:

- Projective limit generalizes the usual set-theoretic product
- Direct limit " " notion of disjoint union.
- Direct limit is the "dual" notion of projective limit. Just reverse all arrows.

Definition :-

- $(I, \leq)$ be an ordered set, $\mathcal{C}$ any category.
- An inductive system in $\mathcal{C}$ based on I is $(\{X_i\}, \psi_{ij})$ where
  - $X_i \in Ob(\mathcal{C})$ $\forall i$,
  - $\forall i \leq j$ we are given $\psi_{ij} \in Hom_{\mathcal{C}}(X_i, X_j)$ (same direction)

So an inductive system is a covariant functor from $\mathcal{I} \longrightarrow \mathcal{C}$.

- The inductive limit or direct limit of the inductive system $(\{X_i\}, \psi_{ij})$ consists of $L = \varprojlim_i X_i$ and $f_i : X_i \longrightarrow L$ such that

$$\begin{tikzcd}
X_i \arrow[d, "\phi_{ij}"'] \arrow[dr, "f_i"] \arrow[drr, bend left, "\alpha_i"] & & \\
X_j \arrow[r, "f_j"'] \arrow[rr, bend right, "\alpha_j"'] & L \arrow[r, dashed, "\exists!"] & Z
\end{tikzcd}$$

To concretely describe the elements of L, it is helpful to have the ordered set to be a directed set.

Directed set is an ordered set $(I, \leq)$ such that for any ~~given~~ $i, j \in I$, there exists $k \in I$ such that $i \leq k$, $j \leq k$.

Elements of L :- Take the disjoint union $\coprod_i X_i$

Go modulo the equivalence relation :-

$$x_i \sim x_j \iff \exists k \ni i \leq k,\ j \leq k \ \& \ \phi_{ik}(x_i) = \phi_{jk}(x_j).$$

Example:-

Let X be a topological space. Fix $x_0 \in X$.

Directed set :- All open subsets of X containing x_0, ordered by reverse inclusion.

$$U \leq V \iff U \supset V.$$

Direct/induction system (of $\mathbb{R}$-vector spaces) :-

$\mathcal{C}(U)$ = all continuous $\mathbb{R}$-valued functions on U.

$U \leq V$, $\phi_{UV} : \mathcal{C}(U) \longrightarrow \mathcal{C}(V)$ is the restriction map.

Direct limit of this system exists and is denoted

$$\mathcal{C}_{x_0} = \varinjlim_{x_0 \in U} \mathcal{C}(U).$$

$\mathcal{C}_{x_0}$ is a $\mathbb{R}$-vector space.

$\mathcal{C}_{x_0}$ is called :- <u>The stalk at x_0 of the "sheaf" of continuous $\mathbb{R}$-valued functions on X</u>

OR

<u>Germs of continuous functions at x_0.</u>

Elements of $\mathcal{C}_{x_0}$ are equivalence classes of pairs (U, f) where $x_0 \in U \underset{\text{open}}{\subset} X$ and $f : U \to \mathbb{R}$ is continuous and the equivalence relation is:-

$$(U, f) \sim (V, g) \iff f|_{U\cap V} = g|_{U \cap V}.$$