Lecture 17

Functoriality of sheaf cohomology.

§1 Let X be a topological space.
Let A be an abelian group.
Let A_X be the constant sheaf on X determined by A.

Recall :- Cohomology of X with coeffs. in A is defined as
$$H^q(X, A) := H^q(X, A_X).$$

In sheaf cohomology, we know the dependence of $H^q(X, \mathcal{F})$ on $\mathcal{F}$; for example, given a short exact seq. $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of sheaves on X we have a long exact sequence $0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{G}) \to \cdots$

Now we fix A, and understand the dependence of $H^q(X,A)$ on X.
We will see that $X \longmapsto H^q(X,A)$ is a contravariant functor from TOP to AB. An important special case is when $A = \mathbb{Z}$.
If X is reasonably nice (for example X = smooth manifold - which is assumed to be locally cpt., T_2, second countable)
then the sheaf cohomology $H^q(X,\mathbb{Z})$ is the same as singular cohomology groups $H^q(X,\mathbb{Z})$; same notation!

§2 Derived functors of f_* :-

Let $f: X \longrightarrow Y$ be a continuous map of topological spaces.
Then $f_* : S_X \longrightarrow S_Y$ is the direct-image functor.

Recall :- $f_*(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$.

Lemma

f_* is a covariant left-exact functor from S_X to S_Y.
(Recall that S_X & S_Y are abelian categories with enough injectives)

- Covariance is obvious:- $\mathcal{F} \longrightarrow \mathcal{G}$ gives $\mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{G}(f^{-1}(V))$ $\forall\, V \underset{\text{open}}{\subseteq} Y$

$$\mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V), \qquad \mathcal{G}(f^{-1}(V)) = f_*\mathcal{G}(V).$$

- Left exact is because:- If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is an exact sequence in S_X, then for all $U \underset{\text{open}}{\subset} X$, the sequence

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \longrightarrow \mathcal{H}(U)$$ is exact.

Take $U = f^{-1}(V)$, $V \underset{\text{open}}{\subset} Y$.

<u>Note</u>:- If $Y = \{*\}$ - a singleton set then $f_*\mathcal{F} = \mathcal{F}(X)$

$\therefore$ Hence f_* generalizes the notion of the global sections functor.

<u>The lemma says that we can consider the right derived functors $R^q f_*$</u> ..

Given $\mathcal{F} \in S_X$

- fix an injective resolution:- $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$
- Consider the complex : $f_*\mathcal{I}^0 \longrightarrow f_*\mathcal{I}^1 \longrightarrow \cdots \longrightarrow f_*\mathcal{I}^n \longrightarrow \cdots$

Then

$$\boxed{R^q f_*(\mathcal{F}) = \frac{\operatorname{Ker}(f_*\mathcal{I}^q \longrightarrow f_*\mathcal{I}^{q+1})}{\operatorname{Im}(f_*\mathcal{I}^{q-1} \longrightarrow f_*\mathcal{I}^q)}}$$

<u>Note</u>:-
- $R^q f_*(\mathcal{F})$ is a sheaf on Y.
- $R^0 f_*(\mathcal{F}) = f_*\mathcal{F}$
- A short exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ in S_X gives the long exact

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H} \to (R^1 f_*\mathcal{F}) \to R^1 f_*(\mathcal{G}) \to R^1 f_*(\mathcal{H}) \to R^2 f_*(\mathcal{F}) \to \cdots$$

- etc ...

§3 Some observations:-

Proposition:- (Closed Embedding)

If A is a closed subset of X and $i : A \hookrightarrow X$ is the inclusion map.

then $i_* : S_A \longrightarrow S_X$ is an exact functor, in particular, $R^q i_*(\mathcal{F}) = 0$ $\forall q \geq 1$, $\forall \mathcal{F}$.

Pf:- Exercise! (This is one of your homework exercises for this week)

Proposition (f^* is exact)

Let $f : X \longrightarrow Y$ be a cont. map of top. spaces.

$f^* : S_Y \longrightarrow S_X$, the inverse image functor, is a covariant exact functor.

Proof:- Recall :- • $f^*\mathcal{G}(U) = \varinjlim_{f(U) \subset V} \mathcal{G}(V)$

• $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)}$ - Inverse image functor is well-behaved on stalks.

If $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is an exact sequence in S_Y

then $\forall y \in Y$, $0 \to \mathcal{G}'_y \to \mathcal{G}_y \to \mathcal{G}''_y \to 0$ is exact.

$\Rightarrow \forall x \in X$, $0 \to \mathcal{G}'_{f(x)} \to \mathcal{G}_{f(x)} \to \mathcal{G}''_{f(x)} \to 0$ "

i.e., $0 \to (f^*\mathcal{G}')_x \to (f^*\mathcal{G})_x \to (f^*\mathcal{G}'')_x \to 0$ "

$\Rightarrow$ $0 \to f^*\mathcal{G}' \to f^*\mathcal{G} \to f^*\mathcal{G}'' \to 0$ ".

Before the next proposition, recall the adjunction formula:-

$$\boxed{\operatorname{Hom}_{S_X}(f^*\mathcal{G}, \mathcal{F}) = \operatorname{Hom}_{S_Y}(\mathcal{G}, f_*\mathcal{F}).}$$

Proposition (f_* maps injectives to injectives)

$\boxed{\begin{array}{l}\text{Let } f: X \longrightarrow Y \text{ be a cont. map. of top. spaces.} \\ \text{Let } \mathcal{I} \text{ be an injective sheaf on } X. \\ \text{Then } f_* \mathcal{I} \text{ is an injective sheaf on } Y.\end{array}}$

Pf: In any abelian category $\mathcal{A}$, an object $I \in ob(\mathcal{A})$ is injective if the functor $Hom_{\mathcal{A}}(-, I) : \mathcal{A} \longrightarrow AB$ is exact.

We want to say that $Hom_{S_Y}(-, f_* \mathcal{I})$ is exact.

But $Hom_{S_Y}(\mathcal{G}, f_* \mathcal{I}) = Hom_{S_X}(f^* \mathcal{G}, \mathcal{I})$

i.e.,

$$\begin{tikzcd} S_Y \arrow[rr, "Hom_{S_Y}(-, f_*\mathcal{I})"] \arrow[dr, "f^*"'] & & AB \\ & S_X \arrow[ur, "Hom_{S_X}(-, \mathcal{I})"'] & \end{tikzcd}$$

composition of exact functors is exact.

Note:- we have the situation:-

$$S_X \xrightarrow{f_*} S_Y \xrightarrow{\Gamma(Y, -)} AB$$

f_* - left exact which maps injectives to injectives

$\Gamma(Y, -)$ is left exact.

In this situation, Grothendieck's spectral sequence gives the derived functors of a composition of left exact functors

§4 Functoriality :- I

Given a cont. map $f : X \longrightarrow Y$

$\forall\, q \geq 0$, $\forall$ sheaves $\mathcal{G}$ on Y we have a homomorphism of abelian groups :-

$$\boxed{H^q(Y, \mathcal{G}) \longrightarrow H^q(X, f^*\mathcal{G}).}$$

Pf:- Start with an injective resolution of $\mathcal{G}$ in S_Y.

$$0 \to \mathcal{G} \to J^0 \to J^1 \to J^2 \to \cdots$$

Take the inverse image via f^* to get an exact sequence :-

$$0 \to f^*\mathcal{G} \to f^*J^0 \to f^*J^1 \to f^*J^2 \to \cdots$$

Now take any injective resolution of $f^*\mathcal{G}$ in S_X and note that we ~~can~~ lift the identity 1 on $f^*\mathcal{G}$ as:-

$$\begin{array}{ccccccccc}
0 & \to & f^*\mathcal{G} & \to & f^*J^0 & \to & f^*J^1 & \to & f^*J^2 \to \cdots \\
 & & \| & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & f^*\mathcal{G} & \to & \mathcal{I}^0 & \to & \mathcal{I}^1 & \to & \mathcal{I}^2 \to \cdots
\end{array}$$

Take global sections over all of X and consider

(*)
$$\begin{array}{ccccccc}
f^*J^0(X) & \to & f^*J^1(X) & \to & f^*J^2(X) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{I}^0(X) & \to & \mathcal{I}^1(X) & \to & \mathcal{I}^2(X) & \to & \cdots
\end{array}$$

Hence we get a map :- $\quad H^q(f^*J^\bullet(X)) \longrightarrow H^q(\mathcal{I}^\bullet(X)) = H^q(X, f^*\mathcal{G}).$

On the other hand,

$$f^*J^n(X) = \varinjlim_{f(X) \subset V} J^n(V) \Rightarrow \exists \text{ map } J^n(Y) \longrightarrow f^*J^n(X)$$

Hence (*) really looks like:-

$$\begin{tikzcd}
J^0(Y) \arrow[r] \arrow[d] & J^1(Y) \arrow[r] \arrow[d] & J^2(Y) \arrow[r] \arrow[d] & \cdots \\
f^*J^0(X) \arrow[r] \arrow[d] & f^*J^0(X) \arrow[r] \arrow[d] & f^*J^1(Y) \arrow[r] \arrow[d] & \cdots \\
\mathcal{J}^0(X) \arrow[r] & \mathcal{J}^1(X) \arrow[r] & \mathcal{J}^2(X) \arrow[r] & \cdots
\end{tikzcd}$$

Hence, we get a map
$$\begin{array}{ccc}
H^q(J^0(Y)) & \longrightarrow & H^q(\mathcal{J}^0(X)) \\
\| & & \| \\
H^q(Y, \mathcal{G}) & & H^q(X, f^*\mathcal{G})
\end{array}$$

i.e.,
$$\boxed{H^q(Y, \mathcal{G}) \longrightarrow H^q(X, f^*\mathcal{G})}$$

§5

Functoriality - II

Theorem

Let $X \xrightarrow{f} Y$ be a continuous map. Then, ~~[illegible]~~ for any abelian group A, f induces a map

$$H^q(f) : H^q(Y, A) \longrightarrow H^q(X, A).$$

i.e., the functor $X \longmapsto H^q(X, A)$ is a contravariant functor from the category TOP of top. spaces & cont. maps to the category AB " abelian groups & group hom.

Pf:- $H^q(Y, A) = H^q(Y, A_Y)$

$H^q(X, A) = H^q(X, A_X)$.

~~It suffices to observe that~~. By functoriality we have a homomorphism

$$H^q(Y, A_Y) \longrightarrow H^q(X, f^* A_Y).$$

Observe that

$$f^* A_Y = A_X$$

The inverse image of a constant sheaf is a constant sheaf.) Proving this is an exercise!

Note: A very special case is cohomology of spaces with integer coefficients :-

$$H^q(X, \mathbb{Z}).$$

$X \longmapsto H^q(X, \mathbb{Z})$ is a contravariant functor.

6 Functoriality - III

Consider the map $H^q(Y, \mathcal{G}) \longrightarrow H^q(X, f^*\mathcal{G})$.

Suppose $\mathcal{F}$ is a sheaf on X, take $\mathcal{G} = f_*\mathcal{F}$.

We have $H^q(Y, f_*\mathcal{F}) \longrightarrow H^q(X, f^* f_* \mathcal{F})$ ——— (i)

$\mathrm{Hom}_{S_X}(f^* f_* \mathcal{F}, \mathcal{F}) = \mathrm{Hom}_{S_Y}(f_*\mathcal{F}, f_*\mathcal{F}) \ni 1$

$\Rightarrow$ $\exists$ natural map $H^q(X, f^*f_*\mathcal{F}) \longrightarrow H^q(X, \mathcal{F})$ ——— (ii)

From (i) & (ii) we get

$$\boxed{H^q(Y, f_*\mathcal{F}) \longrightarrow H^q(X, \mathcal{F}).}$$

Theorem

Let $f: X \longrightarrow Y$ be cont. map.

Let $\mathcal{F}$ - sheaf on X.

Suppose $R^q f_*(\mathcal{F}) = 0 \quad \forall q \geq 1$ Then

$$H^q(Y, f_*\mathcal{F}) \cong H^q(X, \mathcal{F}) \quad \forall q \geq 1$$

Pf:- Let $0 \to \mathcal{F} \to \mathcal{J}^0 \to \mathcal{J}^1 \to \cdots$ be an injective res. of $\mathcal{F}$

$R^q f_* \mathcal{F} = 0 \quad \forall q \geq 1 \Rightarrow$

Apply f_*, $f_*\mathcal{J}^n$ is also injective,

$0 \to f_*\mathcal{F} \to f_*\mathcal{J}^0 \to f_*\mathcal{J}^1 \to \cdots$ is an inj. res. of $f_*\mathcal{F}$.

$$\begin{aligned} H^q(Y, f_*\mathcal{F}) &= H^q(f_*\mathcal{J}^\bullet(Y)) \\ &= H^q(\mathcal{J}^\bullet(f^{-1}(Y))) = H^q(\mathcal{J}^\bullet(X)) = H^q(X, \mathcal{F}). \end{aligned}$$