Lecture # 14.

Sheaf Cohomology : Definition and an example.

§1 Let X be a topological space.

Recall that a sequence of sheaves is exact if at each stage the kernel sheaf is the image sheaf and this is equivalent to the sequence of stalks being exact for all $x \in X$.

Proposition / Definition.

The following are equivalent :-

① The sequence of sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact.

② The sequence of stalks $0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0$ is exact $\forall x \in X$.

③ $\forall U \underset{\text{open}}{\subset} X$

(i) $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ is exact

(ii) $\forall s'' \in \mathcal{F}''(U)$, $\exists$ open cover $U = \bigcup V_i$, $\exists s_i \in \mathcal{F}(U_i)$ such that $s_i \mapsto s''|_{U_i}$.

- Note:- In ③ a section $s'' \in \mathcal{F}''(U)$ need not come from a section of $\mathcal{F}$ over U however, locally we can lift s''.

- Applying ③ to $U = X$, the sequence

  $0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X)$ is exact

  but $\mathcal{F}(X) \longrightarrow \mathcal{F}''(X)$ need not be surjective.

- $\mathcal{F} \longmapsto \mathcal{F}(X)$ is called the <u>functor of (taking) global sections.</u> This is a left exact covariant functor from the category of sheaves of abelian groups on X to the category of abelian groups.

Definition (Sheaf Cohomology).

(i) $H^0(X, \mathcal{F}) = \mathcal{F}(X) =: \Gamma(X, \mathcal{F})$

(ii) $H^\bullet(X, \mathcal{F}) = R^\bullet\Gamma(X, -)(\mathcal{F})$

Sheaf cohomology = right derived functors of the functor of global sections.

- Recall :- To construct right derived functors (of a covariant functor) you need to take injective resolutions. Right now we can ask if that theory makes sense in our context. Soon we will prove that S_X has enough injectives!

- A fundamental property is that given any short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$ of sheaves on X

we get a long exact sequence (in cohomology) of abelian groups:-

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow H^1(X, \mathcal{F}') \longrightarrow H^1(X, \mathcal{F}) \longrightarrow \cdots \quad \text{etc} \ldots$$

§2 Cohomology of the circle:-

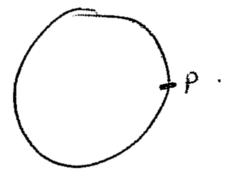
Consider $X = S^1$ – the unit circle. (We will consider a short exact sequence of sheaves on S^1.)

- Let $\mathbb{Z}_X = \mathbb{Z}_{S^1}$ be the constant sheaf on S^1 determined by $\mathbb{Z}$.

(Recall:- $\mathbb{Z}_{S^1}(V) = \{ f : V \longrightarrow \mathbb{Z} \mid f \text{ - locally constant} \} \ \forall\, V \underset{\text{open}}{\subseteq} S^1$.)

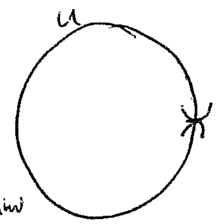
$\forall\, x \in S^1$, the stalk $\mathbb{Z}_{S^1, x} \simeq \mathbb{Z}$.

- Fix one point on S^1. Call it P.



Consider the skyscraper sheaf on S^1 at P with stalk $\mathbb{Z}$. This means the sheaf S_P defined as $S_P(U) = \begin{cases} \mathbb{Z}, & P \in U \\ 0, & P \notin U \end{cases}$

check:- $S_{P,x} = \begin{cases} \mathbb{Z}, & x = P \\ 0, & x \neq P. \end{cases}$



- Let $U = S^1 - \{P\}$

Let $i : U \hookrightarrow S^1$ be inclusion.

Let $\mathbb{Z}_U$ = constant sheaf on U determined by $\mathbb{Z}$.

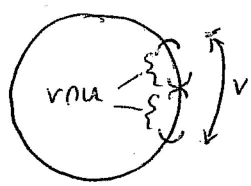
Consider $i_* \mathbb{Z}_U$ = direct image sheaf on S^1 determined by i and $\mathbb{Z}_U$.

Recall:- $(i_* \mathbb{Z}_U)(V) = \mathbb{Z}_U(U \cap V)$
$= \{$ locally constant fns. on $U \cap V$ with values in $\mathbb{Z}$. $\} \ \forall\, V \underset{\text{open}}{\subseteq} S^1$

Compute the stalks of $i_* \mathbb{Z}_U$:-

$$(i_* \mathbb{Z}_U)_x = \begin{cases} \mathbb{Z}_{U,x} = \mathbb{Z} & , \text{ if } x \in U \\ \mathbb{Z} \oplus \mathbb{Z} & , \text{ if } x \notin U \iff x = P. \end{cases}$$

If $x = P$, $\forall$ small nhbds V of x, $V \cap U$ has two pieces:-



- We have a short exact sequence of sheaves:–

$$0 \to \mathbb{Z}_{S^1} \longrightarrow i^* \mathbb{Z}_U \longrightarrow S_p \longrightarrow 0.$$

what are the maps & why is it a short exact sequence?

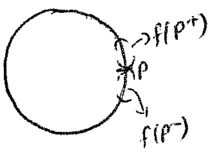
- $\mathbb{Z}_{S^1} \longrightarrow i^* \mathbb{Z}_U$

$$\mathbb{Z}_{S^1}(V) \longrightarrow i^*\mathbb{Z}_U(V) = \mathbb{Z}_U(U \cap V)$$
$$f \longmapsto f|_{U\cap V}.$$

If f is locally const. then $f|_{U\cap V}$ is also "

- $i^* \mathbb{Z}_U \longrightarrow S_p.$

$$i^*\mathbb{Z}_U(V) \longrightarrow S_p(V).$$
$$\| $$
$$\mathbb{Z}_U(U\cap V)$$
$$f \longmapsto \begin{cases} 0 & , p \notin V. \\ f(p^+) - f(p^-) & , p \in V. \end{cases}$$



$$\left.\begin{array}{l} f(p^+) = \text{value of } f \text{ a little above } p \\ f(p^-) = \quad " \quad " \quad " \quad \text{below } p \end{array}\right\} \text{ well-defined by locally-constancy (nice!).}$$

- It is exact because you look at the corresponding sequence of stalks:–

If $x \neq p$,

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}_{S^1,x} & \longrightarrow & i^*\mathbb{Z}_{U,x} & \longrightarrow & S_{p,x} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & \mathbb{Z} & & \mathbb{Z} & & 0. & & \end{array}$$

If $x = p$

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}_{S^1,p} & \longrightarrow & i^*\mathbb{Z}_{U,p} & \longrightarrow & S_{p,p} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \to & \mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & n & \longmapsto & (n,n) & & & & \\ & & & & (a,b) & \longmapsto & a-b & & \end{array}$$

- Let us apply the global sections functor to this sequence :-

$$0 \longrightarrow \mathbb{Z}_{S^1} \longrightarrow i^* \mathbb{Z}_U \longrightarrow \mathcal{S}_p \longrightarrow 0$$

to get

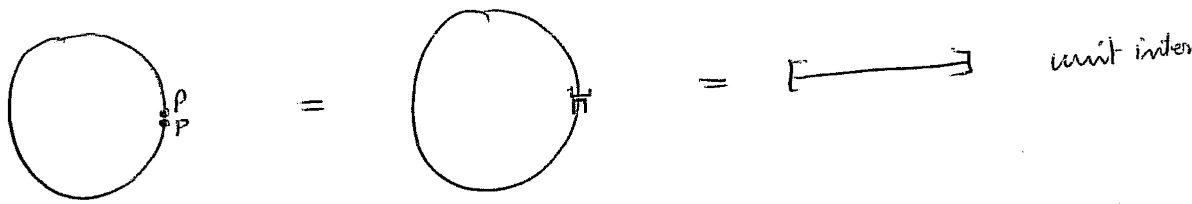
$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}_{S^1}(S^1) & \longrightarrow & i^* \mathbb{Z}_U(S^1) & \longrightarrow & \mathcal{S}_p(S^1) \\ & & \| & & \| & & \| \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z}. \end{array}$$

There is no way this sequence is exact! (Since $\mathbb{Z} \xrightarrow{0} \mathbb{Z}$ is <u>not</u> surjective.)

- Sheaf cohomology would tell us that we have a long exact sequence

$$0 \longrightarrow H^0(S^1, \mathbb{Z}_{S_1}) \longrightarrow H^0(S^1, i_* \mathbb{Z}_U) \longrightarrow H^0(S^1, \mathcal{S}_p) \longrightarrow H^1(S^1, \mathbb{Z}_{S^1}) \longrightarrow H^1(S^1, i_* \mathbb{Z}_U) \cdots$$

- Think of $i_* \mathbb{Z}_U$ as the constant sheaf on the space S^1 with the point p being doubled! But then this space looks like



which we can collapse to a point.

It is believable that $H^1(S^1, i_* \mathbb{Z}_U) = 0$.

- So we are looking at

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0(S^1, \mathbb{Z}) & \longrightarrow & H^0(S^1, i_* \mathbb{Z}_U) & \longrightarrow & H^0(S^1, \mathcal{S}_p) & \longrightarrow & H^1(S^1, \mathbb{Z}) & \longrightarrow 0. \\ & \| & & & & \| & & & \\ & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & ? & \longrightarrow 0 \end{array}$$

? has to be $\mathbb{Z}$.

i.e., $\boxed{H^1(S^1, \mathbb{Z}) = \mathbb{Z}.}$

• One can ask the question if $H^1(S^1, \mathbb{Z}) = \mathbb{Z}$ is a canonical identification?

The answer is no! There are two possible generators of $\mathbb{Z}$.

This is interpreted as there being two possible orientations on S^1 :-

(Later we will see that this is typical. For a compact "oriented" C^∞-manifold M of dim n, $H^n(M, \mathbb{Z}) \simeq \mathbb{Z}$, and choice of a generator is called the fundamental class of M. See section 4.7 of Harder's book.)