Lecture #13.

MANIFOLDS.

§1 Definition :- A topological manifold. X is a Hausdorff space such that $\forall\, x \in X$, $\exists$ open neighbourhood U_x of x, and a homeomorphism $\varphi : U_x \xrightarrow{\sim} U$ where U is an open subset of $\mathbb{R}^n$.

- A topological manifold is a "locally euclidean space".
- If X is connected then it makes sense to talk of dimension of X. (Because, $\underset{\text{open}}{U \subset \mathbb{R}^n}$, $\underset{\text{open}}{V \subset \mathbb{R}^m}$, $U \approx V \Rightarrow n = m$.)
- On a topological manifold we can talk of the sheaf $\mathcal{C}^0_X$ of continuous fu

§2 Definition :- A $\mathcal{C}^\infty$-manifold or a smooth manifold. of dimension n is a topological manifold with a $\mathcal{C}^\infty$-atlas :-

A family $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ such that

(i) U_α is open, $X = \bigcup_\alpha U_\alpha$

(ii) $\varphi_\alpha : U_\alpha \xrightarrow{\sim} U'_\alpha$ a homeomorphism ; U'_α open subset of $\mathbb{R}^n$

(iii) $\forall\, \alpha, \beta$, if $U_\alpha \cap U_\beta \neq \phi$ the maps $\varphi_{\alpha\beta}$ & $\varphi_{\beta\alpha}$ are $\mathcal{C}^\infty$-maps

$$\begin{tikzcd} & \varphi_\alpha(U_\alpha \cap U_\beta) \subset U'_\alpha \arrow[d, shift right, "\varphi_{\alpha\beta}"'] \\ U_\alpha \cap U_\beta \arrow[ur, "\varphi_\alpha"] \arrow[dr, "\varphi_\beta"'] & \\ & \varphi_\beta(U_\alpha \cap U_\beta) \subset U'_\beta \arrow[u, shift right, "\varphi_{\beta\alpha}"'] \end{tikzcd}$$

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}\Big|_{\varphi_\alpha(U_\alpha \cap U_\beta)}$$

- The pair $(U_\alpha, \varphi_\alpha)$ is called a local chart.
- On a smooth manifold X, we have the sheaf $\mathcal{C}^\infty_X$ of smooth functions.

$\forall\, U \underset{\text{open}}{\subset} X, \quad \mathcal{C}^\infty_X(U) = \{ f : U \to \mathbb{R} \mid \text{"} f \text{ is smooth"} \}$

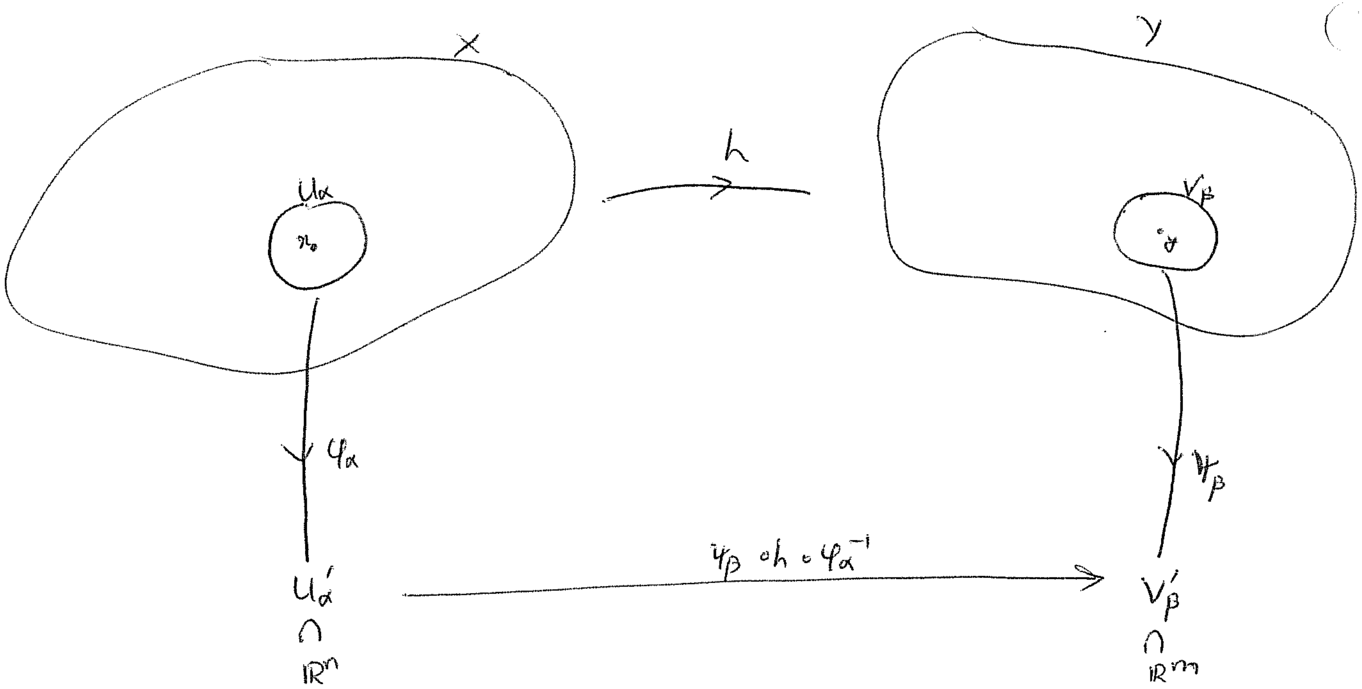
f is smooth is defined locally using charts :-

$\forall\, x \in U$, if $(U_\alpha, \varphi_\alpha)$ is chart with $x \in U_\alpha$ then we require that $f \circ \varphi_\alpha^{-1} : U_\alpha' \to \mathbb{R}$ is infinitely differentiable

$$\begin{tikzcd} & \varphi_\alpha(U_\alpha) = U_\alpha' \underset{\text{open}}{\subset} \mathbb{R}^n. \arrow[d, "f \circ \varphi_\alpha^{-1}"] \\ x \in U_\alpha \arrow[ur, "\varphi_\alpha"] & \\ \cap & \\ U \arrow[r] & \mathbb{R} \end{tikzcd}$$

Because of the local nature of the definition, $\mathcal{C}^\infty_X$ is a sheaf.

Definition :- Let X, Y be smooth manifolds, $h : X \to Y$ is a "smooth map" or a morphism of manifolds. if h is continuous. and $\forall\, x \in X$ if $(U_\alpha, \varphi_\alpha)$ is a chart around x, (V_β, ψ_β) a chart around $y = h(x)$ then the map $\psi_\beta \circ h \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \psi_\beta(V_\beta)$ is a smooth map.



Alternative definition of morphism of manifolds.

The alternative definition uses the sheaves $\mathcal{C}^\infty_X$ and $\mathcal{C}^\infty_Y$. A continuous map $h : X \to Y$ is a morphism of manifolds if $\forall\, U \subset_{\text{open}} X$, $\forall\, V \subset_{\text{open}} Y$, such that $h(U) \subset V$, composition with h induces a map (of $\mathbb{R}$-vector spaces)

$$\begin{aligned} \mathcal{C}^\infty_Y(V) &\xrightarrow{-\circ h} \mathcal{C}^\infty_X(U) \\ f &\longmapsto f\circ h \end{aligned}$$

In other words, $\forall\, U \subset_{\text{open}} X$, we have a map $\underbrace{\varinjlim_{h(U)\subset V} \mathcal{C}^\infty_Y(V)}_{(h^*\mathcal{C}^\infty_Y)(U).} \longrightarrow \mathcal{C}^\infty_X(U)$

Alternative definition:- X, Y be smooth manifolds. $h : X \to Y$ a continuous map is a morphism of manifolds, if composition with h induces a morphism of sheaves $h^*\mathcal{C}^\infty_Y \longrightarrow \mathcal{C}^\infty_X$ $(-\circ h \in \mathrm{Hom}_{S_X}(h^*\mathcal{C}^\infty_Y, \mathcal{C}^\infty_X))$.

§3. Definition:- Complex manifold.

X - locally euclidean, in this case charts $(U_\alpha, \varphi_\alpha)$, $\varphi_\alpha : U_\alpha \to U'_\alpha \subset_{\text{open}} \mathbb{C}^n$.

the patching conditions are in terms of holomorphic functions.

$(\varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha\cap U_\beta)$ is holomorphic)

~~[illegible]~~ Smooth Manifold	$\rightsquigarrow$ Complex manifold.
$\mathbb{R}$	$\mathbb{C}$
Infinitely Diff.	Holomorphic.
$\mathcal{C}^\infty_X$	$\mathcal{O}_X$

- A Riemann surface is a complex manifold of (complex) dimension 1.
- A compact Riemann Surface is a compact complex " " " .

§4 Locally Ringed Spaces.

- If X is a smooth manifold then we call $\mathcal{C}^\infty_X$ - the structure sheaf on X.

  If X is a complex " " " $\mathcal{O}_X$ - " " "

- $x \in X$, $\mathcal{C}^\infty_{X,x}$ = Stalk at x = $\{(U, \varphi) : x \in U \subseteq_{\text{open}} X, \varphi \in \mathcal{C}^\infty_X(U)\}/\sim$ $\quad [\varphi] = [(U,\varphi)]$.

  = germs of smooth functions at x.

  = This is an $\mathbb{R}$-algebra. and has a unique $\underbrace{\text{maximal ideal}}_{m_x}$

  $m_x = \{[\varphi] : \varphi(x) = 0\}$ $\qquad 0 \to m_x \to \mathcal{C}^\infty_{X,x} \xrightarrow{\text{evaluation at } x} \mathbb{R} \to 0$.

  $\therefore$ $\mathcal{C}^\infty_{X,x}$ is a local ring.

- X - complex manifold, $\mathcal{O}_{X,x}$ is a local ring $\qquad 0 \to m_x \to \mathcal{O}_{X,x} \to \mathbb{C} \to 0$

Defn:- ① A *locally ringed space* is a topological space X together with a sheaf $\mathcal{O}_X$ of rings whose stalks are local rings.

② Let (A, m_A) & (B, m_B) be local rings. A homomorphism $\varphi : A \to B$ is said to be a *local homomorphism of local rings*. if $\varphi^{-1}(m_B) = m_A$.

③ Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces. A *morphism of locally ringed spaces* from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where

- $f : X \to Y$ is continuous
- $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ a morphism of sheaves which induces local homomorph at the level of stalks.

(Note:- $\mathrm{Hom}_{S_X}(f^*\mathcal{O}_Y, \mathcal{O}_X) = \mathrm{Hom}_{S_Y}(\mathcal{O}_Y, f_*\mathcal{O}_X)$
So $f^\#$ may as well be thought of a morphism $f^\# : f^*\mathcal{O}_Y \to \mathcal{O}_X$.)

§5 Sheaf-theoretic definition of Manifolds:-

If U is an open subset of $\mathbb{R}^n$ then by $\mathcal{C}^\infty_U$ we mean the sheaf of smooth fns. on U and if $U \underset{\text{open}}{\subseteq} \mathbb{C}^n$, $\mathcal{O}_U$ - sheaf of holomorphic functions on U.

Defn:- A smooth manifold X of dimension n is a locally ringed space $(X, \mathcal{C}^\infty_X)$ which is locally isomorphic to $(U, \mathcal{C}^\infty_U)$ for open subset U of $\mathbb{R}^n$, (i.e., $\forall x \in X$, $\exists$ open U_x, $x \in U_x$, and an isomorphism

$$(U_x, \mathcal{C}^\infty_X|_{U_x}) \simeq (U, \mathcal{C}^\infty_U) \quad \text{for } U \underset{\text{open}}{\subseteq} \mathbb{R}^n,)$$

and for which $\mathcal{C}^\infty_X$ is a subsheaf of the sheaf of continuous functions on X.

Defn:- A complex manifold X of dimension n is a locally ringed space $(X, \mathcal{O}_X)$ such that

- $\mathcal{O}_X$ is a subsheaf of the sheaf. of cont. functions
- Locally, $(X, \mathcal{O}_X)$ looks like $(U, \mathcal{O}_U)$ with $U \underset{\text{open}}{\subseteq} \mathbb{C}^n$.

Defn:- A morphism of (smooth/complex) manifolds. is a morphism of locally ringed spaces.

§6 Examples of manifolds:-

① **Complex projective space**:- $\mathbb{P}^n(\mathbb{C})$:-

$$\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^{n+1} - \{0\} / \mathbb{C}^* = \{(z_0, \dots, z_n) : \begin{array}{l} \text{Not all } z_i \neq 0 \\ (z_0, \dots, z_n) \sim (w_0, \dots, w_n) \text{ if} \\ \exists \lambda \in \mathbb{C}^* \text{ s.t. } z_i = \lambda w_i \end{array} \}.$$

= Space of all lines in $\mathbb{C}^{n+1}$ passing through the origin.

$\mathbb{P}^n(\mathbb{C}) = U_0 \cup U_1 \cup \dots \cup U_n$ $\qquad$ U_i - affine patches

$$U_i = \{[z_0, \dots, z_n] : z_i \neq 0\} \xrightarrow{\sim} \mathbb{C}^n$$

$$[z_0, \dots, z_n] \longmapsto \left(\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_n}{z_i}\right)$$

$\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}$ = structure sheaf on $\mathbb{P}^n(\mathbb{C})$ defined using these affine patches:-

$$\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(U) = \{ f: U \to \mathbb{C} \mid f|_{U\cap U_i} : \underset{\substack{\cap \\ \mathbb{C}^n}}{U \cap U_i} \longrightarrow \mathbb{C} \text{ is holomorphic } \forall i \}$$

For example:-

$$\mathbb{P}^1(\mathbb{C}) = U_0 \cup U_1 \qquad \begin{array}{l} U_0 = \{[1, z] : z \in \mathbb{C}\} \\ U_1 = \{[w, 1] : w \in \mathbb{C}\} \end{array}$$

①′ **Real projective space** $\mathbb{P}^n(\mathbb{R})$

$\mathbb{P}^n(\mathbb{R}) = \mathbb{R}^{n+1} - \{0\} / \mathbb{R}^*$ = All lines in $\mathbb{R}^{n+1}$ through origin.

$= S^n / [\text{antipodal points}]$

$\mathbb{P}^n(\mathbb{R}) = U_0 \cup \dots \cup U_n$, affine patches, $U_i \simeq \mathbb{R}^n$.

$$\mathcal{C}^\infty_{\mathbb{P}^n(\mathbb{R})}(U) = \{ f: U \to \mathbb{R} \mid \forall i,\ f|_{U\cap U_i} : \underset{\substack{\cap \text{ open} \\ \mathbb{R}^n}}{U \cap U_i} \longrightarrow \mathbb{R} \text{ is smooth} \}.$$

(2) $\Omega = \{ n_1 \omega_1 + n_2 \omega_2 : n_1, n_2 \in \mathbb{Z} \} \simeq \mathbb{Z}^2 \subset \mathbb{C}$ a lattice

($\{\omega_1, \omega_2\}$ is an $\mathbb{R}$-basis for $\mathbb{C}$.)

- Consider the projection $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Omega$
- check:- π is a local homeomorphism.
- Topologically, $\mathbb{C}/\Omega$ looks like a 1-torus :-

- $\mathbb{C}/\Omega$ has the structure of a complex manifold of dimension 1 :-

$$\mathcal{O}_{\mathbb{C}/\Omega}(U) = \left\{ f : U \to \mathbb{C} \;\middle|\; \begin{tikzcd} \pi^{-1}(U) \arrow[r] \arrow[d, "\pi"'] & \mathbb{C} \\ U \arrow[ur, "f"'] & \end{tikzcd} \text{ is holomorphic} \right\}.$$

- $\mathbb{C}/\Omega$ is a compact Riemann surface.