

Lecture-1

Categories & Functors.

§1 A Category consists of "objects" & "morphisms" which may sometimes be composed.

• Definition A category \mathcal{C} is

(i) A class of objects $Ob(\mathcal{C})$

(ii) A family of disjoint sets $Hom_{\mathcal{C}}(A, B)$, one for each pair of objects A, B .

(iii) For each triple $A, B, C \in Ob(\mathcal{C})$ a function (called composition)

$$Hom_{\mathcal{C}}(A, B) \times Hom_{\mathcal{C}}(B, C) \longrightarrow Hom_{\mathcal{C}}(A, C)$$

$$(\phi, \psi) \longmapsto \psi \circ \phi$$

(iv) For each $A \in Ob(\mathcal{C})$, a distinguished element $1_A \in Hom_{\mathcal{C}}(A, A)$ (called identity)

such that

Axiom-1 :- Associativity of composition $f \circ (g \circ h) = (f \circ g) \circ h$

Axiom-2 :- $1_B \circ \phi = \phi \circ 1_A = \phi \quad \forall \phi \in Hom_{\mathcal{C}}(A, B)$.

• Note:- $Ob(\mathcal{C})$ is a "class" or "collection" } "class" vs "set"
 $Hom(A, B)$ is a "set"

In a set we have the notion of two elements being equal.

If $Ob(\mathcal{C})$ is a set, then it is called a small category.

• Elements of $Hom_{\mathcal{C}}(A, B)$ are called morphisms or arrows.

• Examples:-

(1) ENS or $SETS$

-- sets and functions.

(2) $VECT_k$

-- vector spaces/ k and k -linear maps

③ $R\text{-MOD}$:- (R is a ring), ~~left~~ left R -modules & R -linear maps.
 $\text{MOD-}R$:- right R -modules & R -linear maps

④ AB :- abelian groups & group homomorphisms.

$$AB = \mathbb{Z}\text{-MOD} = \text{MOD-}\mathbb{Z}.$$

⑤ GROUPS :- groups & group homomorphisms.

AB is a "sub category" of GROUPS .

⑥ TOP :- topological spaces & continuous maps.

⑦ X -topological space gives a category \mathcal{C}_X .

$\text{ob}(\mathcal{C}_X) =$ open subsets of X .

$$\text{Hom}_{\mathcal{C}_X}(U, V) = \begin{cases} \{\phi_{UV}\} & , \quad U \subset V \\ \text{empty set} & , \quad U \not\subset V. \end{cases}$$

⑦' (P, \leq) - partially ordered set. $(x \leq y; \quad x \leq y, y \leq z \Rightarrow x \leq z; \quad x \leq y \leq z \Rightarrow x \leq z)$

gives a category \mathcal{C}_P :-

$$\text{ob}(\mathcal{C}_P) = P, \quad \text{Hom}_{\mathcal{C}_P}(x, y) = \begin{cases} \{\phi_{xy}\} & , \quad x \leq y \\ \text{empty set} & , \quad x \not\leq y. \end{cases}$$

⑧. Let $A =$ category of fin. dim. vector spaces/ k (k -field)

Then \neq notion of equality in $\text{ob}(A)$

Let $B =$ category of framed fin. dim. vector spaces/ k .

$\text{ob}(B) =$ A fin. dim. vector space V/k plus a choice of an ordered basis. $[V] = (V, \{v_1, \dots, v_n\})$

Morphisms are k -linear maps sending a basis to basis while preserving the order.

If $[V] \cong [W]$ then there is a unique isomorphism.

⑨ Every categorical notion has a "dual" notion. Obtained by reversing all arrows!

If \mathcal{C} is a category then $\mathcal{C}^{opp} =$ opposite category.

$$ob(\mathcal{C}^{opp}) = ob(\mathcal{C}).$$

$$Hom_{\mathcal{C}^{opp}}(A, B) := Hom_{\mathcal{C}}(B, A)$$

§2 Functors.

A functor takes you from one category to another while preserving compositions & maps identities to identities.

Defn :- A covariant functor. $F: \mathcal{C} \rightarrow \mathcal{D}$ from a category \mathcal{C} to a category \mathcal{D} is a rule (or really a pair of functions)

$$F: ob(\mathcal{C}) \rightarrow ob(\mathcal{D})$$

$$A \mapsto FA$$

and $Hom_{\mathcal{C}}(A, B) \rightarrow Hom_{\mathcal{D}}(FA, FB)$

$$\varphi \mapsto F\varphi.$$

such that

$$(i) F(1_A) = 1_{FA}$$

$$(ii) F(\varphi \circ \psi) = F\varphi \circ F\psi.$$

(Covariant \equiv directions of arrows are preserved.)

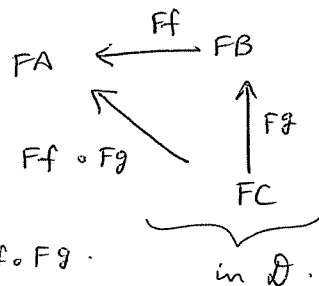
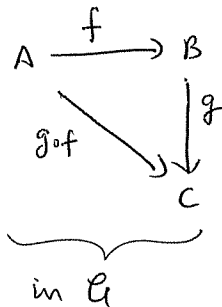
Defn :- A contravariant functor.

$$F: \mathcal{C} \rightarrow \mathcal{D} \dots$$

$$Hom_{\mathcal{C}}(A, B) \rightarrow Hom_{\mathcal{D}}(FB, FA)$$

such that \dots , $F(\varphi \circ \psi) = F\psi \circ F\varphi.$

For example:-



$$\therefore F(g \circ f) = Ff \circ Fg.$$

Examples:-

① Forgetful functor $\mathbb{F}: \text{GROUPS} \rightarrow \text{SETS}$. (covariant)

If G is a group then $\mathbb{F}G$ is the underlying set G .

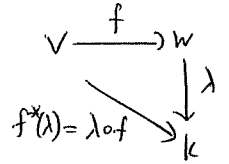
② Duality functor $\text{VECT}_k \rightarrow \text{VECT}_k$ (contravariant).

$$V \mapsto V^* := \text{Hom}_k(V, k).$$

This is the space of all k -linear functionals.

$$f \in \text{Hom}_k(V, W) \mapsto f^* \in \text{Hom}_k(W^*, V^*)$$

$$f^*(\lambda) = \lambda \circ f.$$



③ An example for the "future":-

X - topological space.

A sheaf of abelian groups on X .

$$\mathcal{S}: \mathcal{C}_X \rightarrow \text{AB}$$

is a contravariant functor such that ----- (some properties--)

④ The Hom-functors.

Let \mathcal{C} be any category.

Fix $x \in \text{ob}(\mathcal{C})$.

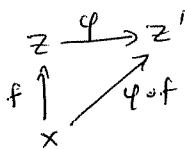
(4a) $h_x: \mathcal{C} \rightarrow \text{SETS}$.

$h_x(Y) = \text{Hom}(X, Y)$.

Given $\varphi \in \text{Hom}(Z, Z')$ we get

$$h_x(\varphi) \in \text{Hom}(X, Z) \xrightarrow{f} \text{Hom}(X, Z')$$

$$f \mapsto \varphi \circ f$$



h_x is often denoted $\text{Hom}_{\mathcal{C}}(X, -)$

h_x is a covariant functor.

(4b) $h_x^{\circ}: \mathcal{C} \rightarrow \text{SETS}$.

$h_x^{\circ} = \text{Hom}(-, X)$

contravariant.

Given $\varphi \in \text{Hom}(Z, Z')$, $h_x^{\circ}(\varphi): \text{Hom}(Z', X) \rightarrow \text{Hom}(Z, X)$

$$f \mapsto f \circ \varphi.$$

