Lecture-1 Categories & Functors.

§1 A category consists of "objects" & "morphisms" which may sometimes be composed.

- Definition: A category $\mathcal{C}$ is
  (i) A class of objects $Ob(\mathcal{C})$
  (ii) A family of disjoint sets $Hom_{\mathcal{C}}(A,B)$, one for each pair of objects A, B.
  (iii) For each triple $A, B, C \in ob(\mathcal{C})$ a function (called composition)

$$Hom_{\mathcal{C}}(A,B) \times Hom_{\mathcal{C}}(B,C) \longrightarrow Hom_{\mathcal{C}}(A,C)$$
$$(\phi, \psi) \longmapsto \psi \circ \varphi$$

  (iv) For each $A \in Ob(\mathcal{C})$, a distinguished element (called identity) $1_A \in Hom_{\mathcal{C}}(A,A)$

such that

Axiom-1 :- Associativity of composition $f \circ (g \circ h) = (f \circ g) \circ h$

Axiom-2 : $1_B \circ \varphi = \varphi \circ 1_A = \varphi \quad \forall \varphi \in Hom_{\mathcal{C}}(A,B)$.

- Note :- $Ob(\mathcal{C})$ is a "class" or "collection" } "Class" Vs "Set"

  $Hom(A,B)$ is a "set"

  In a set we have the notion of two elements being equal.

  If $ob(\mathcal{C})$ is a set then it is called a small category.

- Elements of $Hom_{\mathcal{C}}(A,B)$ are called morphisms. or arrows.

- Examples :-
  ① ENS or SETS :- sets and functions.
  ② $VECT_k$:- vector spaces$/k$ and k-linear maps

(3) R-MOD :- (R is a ring), left R-modules & R-linear maps.

MOD-R :- right R-modules & R-linear maps

(4) AB :- abelian groups & group homomorphisms.

$$\text{AB} = \mathbb{Z}\text{-MOD} = \text{MOD-}\mathbb{Z}.$$

(5) GROUPS :- groups & group homomorphisms.

AB is a "<u>sub category</u>" of GROUPS.

(6) TOP :- topological spaces & continuous maps.

(7) X - topological space. gives a category $\mathcal{C}_X$.

$ob(\mathcal{C}_X)$ = open subsets of X.

$$\mathrm{Hom}_{\mathcal{C}_X}(U,V) = \begin{cases} \{\phi_{UV}\}, & \text{if } U \subset V \\ \text{empty set}, & \text{if } U \not\subset V. \end{cases}$$

(7') $(P, \leq)$ - partially ordered set. ($x \leq x$; $x \leq y, y \leq x \Rightarrow x = y$; $x \leq y \leq z \Rightarrow x \leq z$).

Gives a category $\mathcal{C}_P$:-

$ob(\mathcal{C}_P) = P$, $\quad \mathrm{Hom}_{\mathcal{C}_P}(x,y) = \begin{cases} \{\phi_{xy}\}, & x \leq y \\ \text{empty set}, & x \not\leq y. \end{cases}$

(8). Let $\mathcal{A}$ = category of fin. dim. vector spaces$/k$ (k - field)

Then $\nexists$ notion of equality in $ob(\mathcal{A})$

- Let $\mathcal{B}$ = category of <u>framed</u> fin. dim. vector spaces$/k$.

$ob(\mathcal{B})$ = A fin. dim. vector space V/k plus a choice of an ordered basis. $[V] = (V, \{v_1, \dots, v_n\})$

Morphisms are k-linear maps sending a basis to basis while preserving the order.

If $[V] \simeq [W]$ then there is a <u>unique</u> isomorphism.

$\Rightarrow \exists$ notion of equality in $ob(\mathcal{B})$

(9) Every categorical notion has a "dual" notion. obtained by reversing all arrows!

If $\mathcal{C}$ is a category then $\mathcal{C}^{opp}$ = opposite category.

- $ob(\mathcal{C}^{opp}) = ob(\mathcal{C})$.
- $Hom_{\mathcal{C}^{opp}}(A, B) := Hom_{\mathcal{C}}(B, A)$

§2 Functors.

A functor takes you from one category to another while preserving compositions & maps identities to identities.

<u>Defn</u> :- A <u>covariant functor</u>. $F : \mathcal{C} \longrightarrow \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a rule (or really a pair of functions)

$$F : ob(\mathcal{C}) \longrightarrow ob(\mathcal{D}) \qquad \text{and} \qquad Hom_{\mathcal{C}}(A, B) \longrightarrow Hom_{\mathcal{D}}(FA, FB)$$

$$A \longmapsto FA \qquad\qquad \varphi \longmapsto F\varphi.$$

such that (i) $F(1_A) = 1_{FA}$

(ii) $F(\varphi \circ \psi) = F\varphi \circ F\psi$.

(Covariant $\equiv$ direction of arrows are preserved.)

<u>Defn</u> :- A <u>contravariant functor</u>. $F : \mathcal{C} \longrightarrow \mathcal{D}$ - - -

$$Hom_{\mathcal{C}}(A, B) \longrightarrow Hom_{\mathcal{D}}(FB, FA)$$

- - - - - -

such that - - - - - -, $F(\varphi \circ \psi) = F\psi \circ F\varphi$.

For example:-

$$\underbrace{\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}}_{\text{in } \mathcal{C}} \qquad \underbrace{\begin{tikzcd} FA & FB \arrow[l, "Ff"'] \\ & FC \arrow[u, "Fg"'] \arrow[ul, "Ff\circ Fg"] \end{tikzcd}}_{\text{in } \mathcal{D}}$$

$\therefore F(g \circ f) = Ff \circ Fg$.

Examples :-

① Forgetful functor $\mathcal{F}$: GROUPS $\longrightarrow$ SETS. (covariant)

If G is a group then $\mathcal{F}G$ is the underlying set G.

② Duality functor $\mathrm{VECT}_k \longrightarrow \mathrm{VECT}_k$ (contravariant).

$V \longmapsto V^* := \mathrm{Hom}_k(V, k)$.

This is the space of all k-linear functionals.

$f \in \mathrm{Hom}_k(V, W) \longmapsto f^* \in \mathrm{Hom}_k(W^*, V^*)$

$f^*(\lambda) = \lambda \circ f$.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "f^*(\lambda) = \lambda\circ f"'] & W \arrow[d, "\lambda"] \\ & k \end{tikzcd}$$

③ An example for the "future" :-

X - topological space.

A sheaf of abelian groups on X. is a contravariant functor

$\mathcal{S} : \mathcal{C}_X \longrightarrow AB$ such that ----- (some properties --)

④ The Hom-functors.

Let $\mathcal{C}$ be any category. Fix $X \in \mathrm{Ob}(\mathcal{C})$.

(4a) $\boxed{h_X : \mathcal{C} \longrightarrow \text{SETS}.}$

- $h_X(Y) = \mathrm{Hom}(X, Y)$.
- Given $\varphi \in \mathrm{Hom}(Z, Z')$ we get

$h_X(\varphi) : \mathrm{Hom}(X, Z) \longrightarrow \mathrm{Hom}(X, Z')$

$f \longmapsto \varphi \circ f$

$$\begin{tikzcd} Z \arrow[r, "\varphi"] & Z' \\ X \arrow[u, "f"] \arrow[ur, "\varphi\circ f"'] & \end{tikzcd}$$

h_X is often denoted $\mathrm{Hom}_{\mathcal{C}}(X, -)$

h_X is a covariant functor.

(4b) $\boxed{h_X^\circ : \mathcal{C} \longrightarrow \text{SETS}.}$

- $h_X^\circ = \mathrm{Hom}(-, X)$
- contravariant.

Given $\varphi \in \mathrm{Hom}(Z, Z')$, $h_X^\circ(\varphi) : \mathrm{Hom}(Z', X) \longrightarrow \mathrm{Hom}(Z, X)$

$f \longmapsto f \circ \varphi$.

$$\begin{tikzcd} Z \arrow[r, "\varphi"] \arrow[dr, "f\circ\varphi"'] & Z' \arrow[d, "f"] \\ & X \end{tikzcd}$$