TOPICS IN GEOMETRY: SHEAF THEORY
MATH 6490, SPRING 2009
HOMEWORK 10

In the following exercises X is a topological space, $\mathcal{U} = \{U_i\}$ is an open cover of X, and $\mathcal{F}$ is a sheaf of abelian groups on X.

Exercise 1. Consider the map $d : C^q(X, \mathcal{U}, \mathcal{F}) \to C^{q+1}(X, \mathcal{U}, \mathcal{F})$ defined as in the definition of the Čech complex. Show that $d^2 = 0$. (So we are indeed talking about a complex.)

Exercise 2. Let $C^q_{\text{alt}}(X, \mathcal{U}, \mathcal{F})$ be the alternating cochains in degree q. Show that

(1) $d(C^q_{\text{alt}}(X, \mathcal{U}, \mathcal{F})) \subset C^{q+1}_{\text{alt}}(X, \mathcal{U}, \mathcal{F})$. (That is the alternating Čech complex is a subcomplex of the Čech complex.)

(2) The inclusion $(C^\bullet_{\text{alt}}(X, \mathcal{U}, \mathcal{F}), d) \hookrightarrow (C^\bullet(X, \mathcal{U}, \mathcal{F}), d)$ induces an isomorphism in cohomology.

Exercise 3. Let Δ^n be the usual n-simplex:

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^n \;:\; x_i \geq 0, \text{ and } \sum_i x_i = 1\}.$$

Consider the open cover $\mathcal{U} = \{U_0, \ldots, U_n\}$ of Δ^n where U_i consists of all those points for which $x_i > 0$. Show that

$$\check{H}^q(\Delta^n, \mathcal{U}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases}$$

Exercise 4. Note that $S^n \approx \partial \Delta^{n+1} \subset \mathbb{R}^{n+2}$. Let $V_i = U_i \cap S_n$. Let $\mathcal{V} = \{V_i\}$. Show that

$$\check{H}^q(S^n, \mathcal{V}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \text{ or } q = n, \\ 0 & \text{if } q \notin \{0, n\}. \end{cases}$$

Exercise 5. Let $\{\mathcal{F}_i\}$ be a collection of sheaves of abelian groups on X. Define a presheaf $\prod_i \mathcal{F}_i$ by $U \mapsto \prod_i \mathcal{F}_i(U)$. Show that this presheaf is in fact a sheaf, and is the categorical product of the sheaves $\{\mathcal{F}_i\}$. Assuming that for each $x \in X$ one has $\mathcal{F}_{i,x} = 0$ for all but finitely many i, show that $(\prod_i \mathcal{F}_i)_x = \prod_i \mathcal{F}_{i,x}$.

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