TOPICS IN GEOMETRY: SHEAF THEORY
MATH 6490, SPRING 2009
HOMEWORK 1

Let $\mathcal{C}$ be a category. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is a category such that

(1) Every object of $\mathcal{A}$ is an object of $\mathcal{C}$, i.e., $\mathrm{Ob}(\mathcal{A}) \subset \mathrm{Ob}(\mathcal{C})$.
(2) Every morphism in $\mathcal{A}$ is a morphism of $\mathcal{C}$, i.e, $\mathrm{Hom}_{\mathcal{A}}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B)$.
(3) The rule for composition of morphisms in $\mathcal{A}$ is the rule for composition in $\mathcal{C}$.

We say $\mathcal{A}$ is a *full subcategory* of $\mathcal{C}$ if it is a subcategory such that for all objects A and B in $\mathcal{A}$ one has $\mathrm{Hom}_{\mathcal{A}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Exercise 1. Give an example of a category and a full subcategory. Give an example of a category and a subcategory which is not full.

A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *faithful* (resp. *full*) if for all objects A, B of $\mathcal{C}$, the map $f \mapsto Ff$ from $\mathrm{Hom}_{\mathcal{C}}(A, B)$ to $\mathrm{Hom}_{\mathcal{D}}(FA, FB)$ is injective (resp. surjective). The same definition applies to a contravariant functor, except the map is from $\mathrm{Hom}_{\mathcal{C}}(A, B)$ to $\mathrm{Hom}_{\mathcal{D}}(FB, FA)$. A functor is said to be *fully faithful* if it is both full and faithful.

Exercise 2. (1) Give two examples of covariant functors. For each example, decide if it is full and/or faithful.
(2) Give two examples of contravariant functors. For each example, decide if it is full and/or faithful.

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Let F and G be two covariant functors from $\mathcal{C}$ to $\mathcal{D}$. We say $\eta : F \to G$ is a *natural transformation* if for each $A \in \mathrm{Ob}(\mathcal{C})$ we are given $\eta_A \in \mathrm{Hom}_{\mathcal{D}}(FA, GA)$ such that for all objects A, B of $\mathcal{C}$ and all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

If η_A is an isomorphism for all A then we say F and G are *naturally isomorphic.*

Exercise 3. Let $\mathcal{V}$ stand for the category whose objects are finite-dimensional vector spaces over a field F, and whose morphisms are F-linear maps. Show that the identity functor on $\mathcal{V}$ is naturally isomorphic to the *double-dual* functor from $\mathcal{V}$ to itself. (The double-dual functor assigns to every $V \in \mathrm{Ob}(\mathcal{V})$ its double-dual V^{**}; recall that a linear functional on V is a linear map $\ell : V \to F$, and the dual V^* of V consists of all linear functionals on V and is written as $V^* = \mathrm{Hom}_F(V, F)$.)

Exercise 4. Let **GROUP** be the category of all groups and group homomorphisms, and let **AB** be the subcategory of all abelian groups. Let G and H be abelian groups.

(1) Prove that the product of G and H in **AB** is the same as the product in **GRP**.
(2) Explain why the coproduct of G and H in **AB** need not be same as the coproduct in **GRP**. Give an example.

In the following problem, $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$ which is the inverse limit over n of all the $\mathbb{Z}/n$. For a prime p, the group of p-adic integers is denoted $\mathbb{Z}_p$ which is the inverse limit over n of all $\mathbb{Z}/p^n$.

Exercise 5. Show that

(1) $\widehat{\mathbb{Z}}$ is uncountable.
(2) $\widehat{\mathbb{Z}} \simeq \prod_p \mathbb{Z}_p$. (Hint: Use the chinese remainder theorem.)

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