CW-complexes & cohomology of $\mathbb{P}^n(\mathbb{C})$.

§1 CW-complexes

A CW-complex is a space built out of "cells" in a nice way.

C - closure finite

W - weak topology

Construction

Let $K^{(0)}$ = discrete set of points. These are the 0-cells.

Assume $K^{(n-1)}$ is defined.

Consider a set of disks D^n_σ as σ ranges over some indexing set.

Let $Y = \amalg D^n_\sigma$ - disjoint union.

Let $\{f_{\partial\sigma}\}$ be a a collection of maps $f_{\partial\sigma} : \partial D^n_\sigma = S^{n-1} \longrightarrow K^{(n-1)}$.

$B = \amalg \partial D^n_\sigma \subset Y$ then we have $f : B \longrightarrow K^{(n-1)}$.

Namely, $f = \amalg f_{\partial\sigma}$

Define $K^{(n)} = K^{(n-1)} \cup_f Y$.

Let $K = \bigcup_{n \geq 0} K^{(n)}$.

Put a topology on K as:- $U \subset K$ is open $\Leftrightarrow$ $U \cap K^{(n)}$ is open in $K^{(n)}$ $\forall n \geq$

This is called the "weak" topology".

Such a space K is called a CW-complex.

- Review of attaching spaces:-

Given a situation like:

$$\begin{array}{ccc} X & & \\ \uparrow & & \\ A & \xrightarrow{f} & Y \end{array}$$

we can consider $Y \cup_f X$.

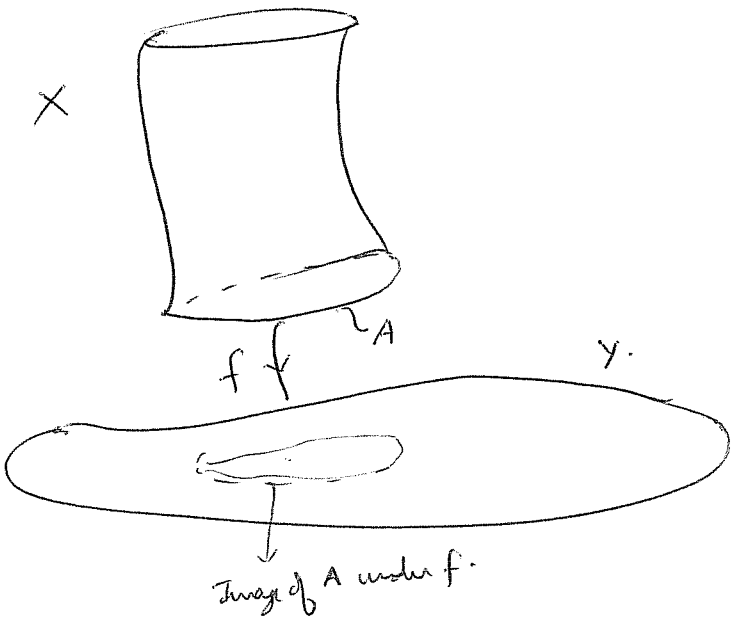
which is defined as:

$$Y \cup_f X = \frac{Y \amalg X}{\sim}$$

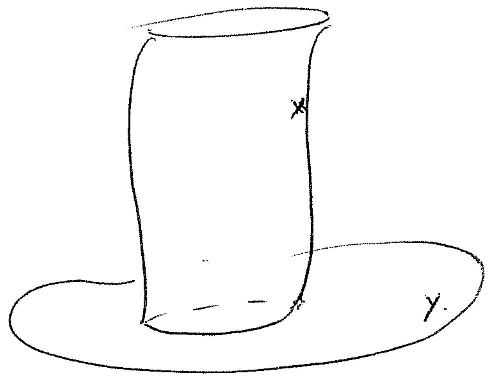
$\sim$ = equivalence generated by $a \sim f(a)$

$\therefore$ If $u, v \in Y \amalg X$ then $u \sim v \iff \begin{cases} \cdot\ u = v \\ \cdot\ u, v \in A, \quad f(u) = f(v) \\ \cdot\ u \in A,\ v \in Y, \quad f(u) = v. \end{cases}$

You can visualize this as:-



$Y \cup_f X$ =



Easy exercises:-

(i) $Y \hookrightarrow Y \cup_f X$ is a closed embedding

(ii) $X \setminus A \hookrightarrow Y \cup_f X$ is an open embedding.

§2 To compute the cohomology of a CW-complex.

We do this inductively.

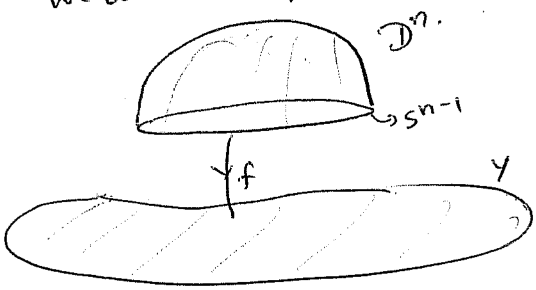
Suppose X is obtained from Y by attaching a single cell: D^n;

i.e., Y is a space

$\partial f : \partial D^n = S^{n-1} \longrightarrow Y$. is a continuous map.

Then $X = Y \cup_f D^n$.

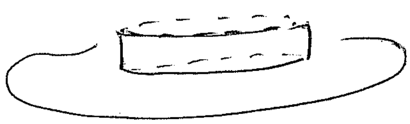
So we are looking at



$X = Y \cup_f D^n$

We will apply the Mayer–Vietoris sequence: for $X = U \cup V$ where.

$U =$



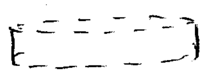
$U = Y \cup_f D^n_\varepsilon$ for some positive $\varepsilon > 0$

$D^n_\varepsilon = \{ (x_1, \dots, x_n) \mid \sum x_i^2 > 1 - \varepsilon \}$

$V = D^n \setminus S^{n-1} \hookrightarrow X$.

$U \cup V = X$

$U \cap V =$



$=$ image of $\{ (x_1, \dots, x_n) \mid 1 - \varepsilon < \sum x_i^2 < 1 \}$

Note:- . U is homotopic to Y. (we can collapse D^n_ε to S^{n-1})

. $U \cap V$ " " S^{n-1} (we can collapse to $\sum x_i^2 = 1 - \frac{\varepsilon}{2}$.)

Mayer-Vietoris sequence gives

$$\cdots \longrightarrow H^q(U\cup V, \mathbb{Z}) \longrightarrow H^q(U,\mathbb{Z}) \oplus H^q(V,\mathbb{Z}) \longrightarrow H^q(U\cap V, \mathbb{Z}) \longrightarrow H^{q+1}(U\cup V, \mathbb{Z}) \longrightarrow$$

$H^q(U\cup V, \mathbb{Z}) = H^q(X, \mathbb{Z})$ $\quad X = U\cup V$

$H^q(U, \mathbb{Z}) = H^q(Y, \mathbb{Z})$ since $U \simeq Y$. (homotopy axiom)

$H^q(V, \mathbb{Z}) =$ ~~$H^q(S^{n-1}, \mathbb{Z})$ since $V \simeq S^{n-1}$~~

$= H^q(D^n \setminus S^{n-1}, \mathbb{Z})$

$= H^q(\mathrm{Int}(D^n), \mathbb{Z}) = \begin{cases} 0, & q \geq 1 \\ \mathbb{Z}, & q = 0. \end{cases}$ $\quad$ $\mathrm{Int}(D^n)$ is contractible

$H^q(U\cap V, \mathbb{Z}) = H^q(S^{n-1}, \mathbb{Z})$, $\quad U\cap V \simeq S^{n-1}$ (homotopy axiom)

$= \begin{cases} \mathbb{Z}, & q = 0, n-1 \\ 0, & q \notin \{0, n-1\} \end{cases}$

Hence we have a sequence that looks like:

$$0 \longrightarrow H^0(X,\mathbb{Z}) \longrightarrow H^0(Y,\mathbb{Z}) \oplus \mathbb{Z} \longrightarrow \underset{\mathbb{Z}}{\overset{\|}{H^0(S^{n-1},\mathbb{Z})}} \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^q(X,\mathbb{Z}) \longrightarrow H^q(Y,\mathbb{Z}) \longrightarrow H^q(S^{n-1},\mathbb{Z}) \longrightarrow H^{q+1}(X,\mathbb{Z}) \longrightarrow \cdots$$

So, if we know the cohomology of Y then we know the cohomology of X.

§3 $\mathbb{P}^n(\mathbb{C})$ – complex projective space.

- $$\mathbb{P}^n(\mathbb{C}) = \frac{\mathbb{C}^{n+1} - \{0\}}{\mathbb{C}^*} = \text{All lines through the origin in } \mathbb{C}^{n+1}.$$

$$= \frac{\{(z_0, \dots, z_n) \mid z_i \in \mathbb{C}\}}{(z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n) \text{ for any } \lambda \in \mathbb{C}^*}$$

So any point in $\mathbb{P}^n(\mathbb{C})$ is represented in "homogeneous coordinates" as $[z_0 : \dots : z_n]$ where $[z_0 : \dots : z_n] = [\lambda z_0 : \dots : \lambda z_n]$.

- $$\boxed{\mathbb{P}^0(\mathbb{C}) = \mathbb{C} - \{0\}/\mathbb{C}^* = \text{just a point}}$$

- $\mathbb{P}^1(\mathbb{C}) = \{[z_0 : z_1]\}$
  - $z_1 = 0 \Rightarrow z_0 \neq 0 \Rightarrow [z_0 : 0] = [1 : 0]$ – A point
  - $z_1 \neq 0 \Rightarrow [z_0 : z_1] = [z_0/z_1 : 1] \simeq \mathbb{C}$.

$\therefore$ $\boxed{\mathbb{P}^1(\mathbb{C}) = \text{Take } \mathbb{C} \text{ and add a "point at infinity"} \approx S^2.}$

This may also be interpreted as :

$\mathbb{P}^1(\mathbb{C}) = \mathbb{C}$ and $\mathbb{P}^0(\mathbb{C})$ – at infinity.

- $\mathbb{P}^n(\mathbb{C}) = \{[z_0 : \dots : z_n]\}$
  - $z_n = 0$, $[z_0 : \dots : z_{n-1}] \in \mathbb{P}^{n-1}(\mathbb{C})$.
  - $z_n \neq 0$, $[z_0 : \dots : z_n] = \left[\frac{z_0}{z_n} : \dots : \frac{z_{n-1}}{z_n} : 1\right] \approx \mathbb{C}^n$

$\therefore$ $$\boxed{\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^n \sqcup \mathbb{P}^{n-1}(\mathbb{C})}$$

We can now describe a CW-complex structure for $\mathbb{P}^n(\mathbb{C})$.

Note :- $\mathbb{C}^n \approx \mathrm{Int}(D^{2n})$.

$$D^{2n} = \{ (a_1, \dots, a_{2n}) \in \mathbb{R}^{2n} \mid \sum a_i^2 \le 1 \}$$

$$= \{ (x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n} \mid \sum x_i^2 + y_i^2 \le 1 \} \quad \text{– Just relabelling.}$$

$$= \{ (z_0, \dots, z_{n-1}) \in \mathbb{C}^n \mid \sum |z_i|^2 \le 1 \}.$$

$$\mathrm{Int}(D^{2n}) = \{ (z_0, \dots, z_{n-1}) \in \mathbb{C}^n \mid \sum |z_i|^2 < 1 \}.$$

<u>Note</u>:- The map

$$\mathrm{Int}(D^{2n}) \longrightarrow \mathbb{C}^n. \qquad \|z\|^2 = \sum |z_i|^2$$

$$(z_0, \dots, z_{n-1}) \longmapsto \left(\frac{z_0}{\sqrt{1 - \|z\|^2}}, \dots, \frac{z_{n-1}}{\sqrt{1 - \|z\|^2}} \right)$$

is a homeomorphism !!

(This is also called "Stereographic projection".)

- Define $f : D^{2n} \longrightarrow \mathbb{P}^n(\mathbb{C})$ by

$$f(z_0, \dots, z_{n-1}) = [z_0 : \dots : z_{n-1} : \sqrt{1 - \sum |z_i|^2}\,]$$

Then $\partial f(\partial D^{2n} = S^{2n-1}) \subset [z_0 : \dots : z_{n-1} : 0] \doteq \mathbb{P}^{n-1}(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$.

The map $D^{2n} \sqcup \mathbb{P}^{n-1}(\mathbb{C}) \longrightarrow \mathbb{P}^n(\mathbb{C})$ factors as

$$\begin{tikzcd}
D^{2n} \sqcup \mathbb{P}^{n-1}(\mathbb{C}) \arrow[r] \arrow[d] & \mathbb{P}^n(\mathbb{C}) \\
D^{2n} \cup_{\partial f} \mathbb{P}^{n-1}(\mathbb{C}) \arrow[ur] &
\end{tikzcd}$$

<u>Ex</u>:- $D^{2n} \cup_{\partial f} \mathbb{P}^{n-1}(\mathbb{C}) \longrightarrow \mathbb{P}^n(\mathbb{C})$ is a homeomorphism.

(Useful detail: $\mathbb{P}^n(\mathbb{C})$ is compact!)

- $\boxed{\mathbb{P}^n(\mathbb{C}) \text{ is obtained from } \mathbb{P}^{n-1}(\mathbb{C}) \text{ by attaching a } 2n\text{-cell.}}$

$\Rightarrow$ $\mathbb{P}^n(\mathbb{C})$ consists of one cell in every even dimension $\le 2n$.

and does not have any odd-dimensional cell.

Cohomology of $\mathbb{P}^n(\mathbb{C})$. :-

Theorem:-

$$\boxed{H^q(\mathbb{P}^n(\mathbb{C}), \mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0, 2, 4, \dots, 2n. \\ 0, & \text{If not, i.e., } q \notin \{0, 2, 4, \dots, 2n\}. \end{cases}}$$

Proof:- Proof is by induction on n:

$$H^q(P^0(\mathbb{C}), \mathbb{Z}) = H^q(pt, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0 \\ 0, & \text{if not} \end{cases}$$

$$H^q(\mathbb{P}^1(\mathbb{C}), \mathbb{Z}) = H^q(S^2, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0, 2 \\ 0, & \text{if not.} \end{cases}$$

Assume $n \geq 1$. Assume theorem is true for $\mathbb{P}^n(\mathbb{C})$.

$\mathbb{P}^{n+1}(\mathbb{C})$ is connected (why?) $\Rightarrow H^0(\mathbb{P}^{n+1}(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}$.

Since $\mathbb{P}^{n+1}(\mathbb{C}) = \mathbb{P}^{n}(\mathbb{C}) \cup_{\partial f} D^{2n+2}$, by the Mayer-Victoris sequence,

we have an exact sequence:

$$\cdots \to H^{q-1}(S^{2n+1}, \mathbb{Z}) \to H^q(\mathbb{P}^{n+1}(\mathbb{C}), \mathbb{Z}) \to H^q(\mathbb{P}^n(\mathbb{C}), \mathbb{Z}) \to H^q(S^{2n+1}, \mathbb{Z}) \to H^{q+1}(\mathbb{P}^{n+1}(\mathbb{C}), \mathbb{Z}) \to \cdots$$

$\therefore$ If $q \notin \{2n+1, 2n+2\}$ then $H^q(S^{2n+1}, \mathbb{Z}) = H^q(S^{2n+1}, \mathbb{Z}) = 0$

and $q \geq 2$

$$\Rightarrow H^q(\mathbb{P}^{n+1}(\mathbb{C}), \mathbb{Z}) = H^q(\mathbb{P}^n(\mathbb{C}), \mathbb{Z})$$

The only remaining cases are $q = 1$, $2n+1$ and $2n+2$.

$\underline{q=1}$:-

$$\begin{array}{ccccccccc} 0 \to & H^0(\mathbb{P}^{n+1}(\mathbb{C})) & \to & H^0(\mathbb{P}^n(\mathbb{C})) \oplus \mathbb{Z} & \to & H^0(S^{2n+1}) & \to & H^1(\mathbb{P}^{n+1}) \to & H^1(\mathbb{P}^n) \to \cdots \\ & \| & & \| & & \| & & & \| \\ & \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} & & & 0. \\ & a & \longmapsto & (a, a) & & & & & \\ & & & (a, b) & \longmapsto & a - b & & & \end{array}$$

$$\Rightarrow H^1(\mathbb{P}^{n+1}(\mathbb{C}), \mathbb{Z}) = 0.$$

$\underline{q = 2n+1}$:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{2n}(S^{2n+1}) \arrow[r] \arrow[d, phantom, "\parallel"] & H^{2n+1}(\mathbb{P}^{n+1}) \arrow[r] & H^{2n+1}(\mathbb{P}^n) \arrow[r] \arrow[d, phantom, "\parallel"] & \cdots \\
 & 0 & & 0 &
\end{tikzcd}$$

$$\Rightarrow \quad H^{2n+1}(\mathbb{P}^{n+1}(\mathbb{C}), \mathbb{Z}) = 0.$$

$\underline{q = 2n+2}$

$$\begin{tikzcd}
\cdots \arrow[r] & H^{2n+1}(\mathbb{P}^n) \arrow[r] \arrow[d, phantom, "\parallel"] & H^{2n+1}(S^{2n+1}) \arrow[r] \arrow[d, phantom, "\parallel"] & H^{2n+2}(\mathbb{P}^{n+1}) \arrow[r] & H^{2n+2}(\mathbb{P}^n) \arrow[r] \arrow[d, phantom, "\parallel"] & \cdots \\
 & 0 & \mathbb{Z} & & 0 &
\end{tikzcd}$$

$$\Rightarrow \quad H^{2n+2}(\mathbb{P}^{n+1}, \mathbb{Z}) = \mathbb{Z}.$$

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