QUADRATIC RECIPROCITY FOR ROOT NUMBERS OF GL(2)

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ABSTRACT. Let F be a local/global field. Let E and K be quadratic semisimple F–algebras. Let ϕ and λ be characters/grössencharacters of $E^\times$ and $K^\times$. We define a local/global symbol $\left(\frac{\phi}{\lambda}\right)$ to be essentially the root number of the representation $\mathrm{BC}_{K/F}(\mathrm{AI}_{E/F}(\phi)) \otimes \lambda^{-1}$. In the spirit of quadratic reciprocity, we prove that

$$\left(\frac{\phi}{\lambda}\right)\left(\frac{\lambda}{\phi}\right) = \phi(-1)\lambda(-1).$$

We then derive some consequences of this reciprocity for characters of supercuspidal representations of GL(2).

CONTENTS

1. INTRODUCTION AND STATEMENTS OF THEOREMS

Let F be a characteristic zero local field. Let π be an irreducible admissible infinite dimensional representation of $\mathrm{GL}_2(F)$. Let K be a quadratic extension of F. Then K^* embeds into $\mathrm{GL}_2(F)$. Let λ be a character of K^* such that $\lambda|_{F^*} = \omega_\pi$, the central character of π. If F is non-Archimedean then Tunnell [15] and Saito [13] have proved that λ occurs in $\pi|_{K^*}$ (i.e., $\mathrm{Hom}_{K^*}(\pi|_{K^*}, \lambda) \neq (0)$) if and only if $\varepsilon(\mathrm{BC}_{K/F}(\pi) \otimes \lambda^{-1})\omega_\pi(-1) = 1$. Here $\mathrm{BC}_{K/F}(\pi)$ is the base change of π to an irreducible representation of $\mathrm{GL}_2(K)$, and $\varepsilon(\mathrm{BC}_{K/F}(\pi) \otimes \lambda^{-1})$ is the root number of the representation $\mathrm{BC}_{K/F}(\pi) \otimes \lambda^{-1}$. Tunnell proved this when the residue characteristic is odd and Saito gave a uniform proof.

Given this result of Tunnell and Saito, we consider the following question. Suppose E is also a quadratic extension of F which may or may not be equal to K and if ϕ is a

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character of E^* and $\pi(\phi) = \mathrm{AI}_{E/F}(\phi)$ is the representation of $\mathrm{GL}_2(F)$ attached to ϕ, then what is the relationship between λ occurring in $\pi(\phi)$ and ϕ occurring in $\pi(\lambda)$? It turns out that we may consider quadratic extensions as well as split quadratic algebras $F \times F$ on the same footing. Also such questions can be asked for Archimedean local fields as well, thus allowing us to consider some global issues on similar lines. For the time being we continue with our local discussion.

The following two examples are prototypical of the question alluded to above and our reciprocity theorem gives a simultaneous explanation of these examples.

Example 1.1. Let $F = \mathbb{R}$ and let $E = \mathbb{C}$. For an integer m, let χ_m be the character of $\mathbb{C}^*$ given by $\chi_m(z) = (z/|z|)^m$. If $m \geq 1$, then $\pi(\chi_m)$ is the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of lowest weight $m+1$. The K-types of $\pi(\chi_m)$ are well known. The character χ_k occurs in $\pi(\chi_m)$ if and only if $k = \pm(m+1 \pm 2r)$ for $r \geq 0$. Hence, for integers $n, m \geq 1$, if χ_n occurs $\pi(\chi_m)$ then χ_m cannot occur in $\pi(\chi_n)$.

Example 1.2. Let F be a non-Archimedean local field of characteristic zero. Let E be a quadratic extension of F and let $K = F \times F$. Let ϕ be a $\mathrm{Gal}(E/F)$-regular character of E^*. Let $\lambda = \lambda_1 \otimes \lambda_2$ be a character of $F^* \times F^*$. We let $\lambda' = \lambda_1\omega_{E/F} \otimes \lambda_2$ where $\omega_{E/F}$ is the character of F^* associated to the extension E/F via local class field theory. Then the following three statements are equivalent:

1. $\phi|_{F^*} = \lambda_1\lambda_2\omega_{E/F}$.
2. λ occurs in $\pi(\phi)|_{F^* \times F^*}$.
3. ϕ occurs in $\pi(\lambda')|_{E^*}$.

The equivalence of (1) and (2) is due to Waldspurger [16] and is an easy consequence of Kirillov theory. The equivalence of (1) and (3) is due to Tunnell [15].

The best way to explain such dichotomy of characters of maximal tori occurring in representations of GL_2, motivated by the results of Tunnell and Saito, seems to be in terms of root numbers. These examples above can be explained in terms of a certain reciprocity property of these root numbers. This is the main theme of this paper.

We now state our results more precisely. Let F be any local field of characteristic zero. Let E and K be quadratic semisimple F–algebras. (A quadratic semisimple F–algebra in simply a quadratic extension in the non-split case and in the split case, is isomorphic to $F \times F$.) Let ϕ be a character of $E^\times$ and let λ be a character of $K^\times$. We fix, for once and for all, a non-trivial additive character $\psi = \psi_F$ of F and for any extension L of F, we let $\psi_L = \psi_F \circ T_{L/F}$ where $T_{L/F}$ is the trace map from L to F.

For a representation π of $\mathrm{GL}_2(F)$ we let $\mathrm{BC}_{E/F}(\pi)$ to be the base change lift of π to a representation of $\mathrm{GL}_2(E)$. If E is split, this is simply $\pi \otimes \pi$ as a representation of $\mathrm{GL}_2(E) = \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. If E is not split then the existence of this lifting is due to Langlands [9]. For a character ϕ of $E^\times$, we let $\pi(\phi) = \mathrm{AI}_{E/F}(\phi)$ be the local automorphic induction of ϕ to an irreducible representation of $\mathrm{GL}_2(F)$. In the non-split case, if ϕ is regular then $\pi(\phi)$ is the supercuspidal representation associated to ϕ, while in the split case, if $\phi = \phi_1 \otimes \phi_2$ and $\phi_1\phi_2^{-1} \neq |\cdot|_F^{\pm}$ it is the principal series representation $\pi(\phi_1, \phi_2)$. We can now define our local symbol.

Definition 1.3 (Local symbol)**.** *Let F be a local field of characteristic zero. Let E and K be quadratic semisimple F–algebras. Let ϕ and λ be characters of $E^\times$ and $K^\times$ respectively. We define the local symbol ϕ over λ as:*

$$\left(\frac{\phi}{\lambda}\right) = \varepsilon(\mathrm{BC}_{K/F}(\pi(\phi)) \otimes \lambda^{-1}, \psi_K)\omega_{\pi(\phi)}(-1).$$

We remark that in the papers of Tunnell and Saito, the right hand side is considered for only those λ, such that λ restricted to F^* is the central character of $\pi(\phi)$. However, we have no such restriction, and so the local symbol need not be ± 1 but is some complex number of absolute value 1. In the spirit of quadratic reciprocity, our main local result is the following theorem.

Theorem 1.4 (Local reciprocity)**.** *Let F be a local field of characteristic zero. Let E and K be quadratic semisimple F–algebras. Let ϕ and λ be characters of $E^\times$ and $K^\times$ respectively. Then*

$$\left(\frac{\phi}{\lambda}\right)\left(\frac{\lambda}{\phi}\right) = \phi(-1)\lambda(-1).$$

The local reciprocity theorem is proved in Section 2. It is a consequence of several standard properties of epsilon factors in the p-adic case, and in the Archimedean case it follows from explicit formulas for epsilon factors.

As a first application of the local reciprocity theorem, we give a simultaneous explanation of the examples above. As a second application, which is more serious, we complement results in Tunnell [15] about characters of supercuspidal representations. We can give a reasonably complete picture of characters of E^* occurring in a supercuspidal representation π of $\mathrm{GL}_2(F)$, if the Langlands parameter of π is obtained by induction from a character of E^*. (See Propositions 3.6, 3.7 and 3.8.)

Our simultaneous considerations of both split and non-split cases, as well as letting the local field be Archimedean, allows us to define a global symbol and prove a corresponding global theorem. Let F now stand for a number field. Let E and K be quadratic semisimple F–algebras. Let ϕ and λ be grössencharacters of E and K respectively, i.e., they are continuous characters of $E^*\backslash\mathbb{A}_E^\times$ and $K^*\backslash\mathbb{A}_K^\times$ respectively.

Definition 1.5 (Global symbol)**.** *With the global notations as above, we define the global symbol of ϕ over λ as*

$$\left(\frac{\phi}{\lambda}\right) = \varepsilon(\mathrm{BC}_{K/F}(\pi(\phi)) \otimes \lambda^{-1}).$$

As can be expected the global symbol factorizes as a product of local symbols and the local reciprocity implies the following global reciprocity theorem.

Theorem 1.6 (Global reciprocity)**.** *With global notations as above,*

$$\left(\frac{\phi}{\lambda}\right)\left(\frac{\lambda}{\phi}\right) = 1.$$

We end this introduction on a speculative note. We believe that the global reciprocity theorem should have implications about (cuspidal) automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ which are distinguished by characters of maximal tori. We also believe that the classical quadratic reciprocity theorem should be a special case of the global reciprocity theorem.

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2. Proof of the local reciprocity theorem

In this section F will denote a local field of characteristic zero. If F is non-Archimedean then we let $\mathcal{O}_F$ denote the ring of integers in F, $\mathfrak{P}_F$ the maximal ideal in $\mathcal{O}_F$ and ϖ_F a uniformizer for F, q_F denote the cardinality of the residue field $k_F = \mathcal{O}_F/\mathfrak{P}_F$ and v_F the additive valuation of F.

Let W_F denote the Weil group of F and let $W'_F = W_F \times SL_2(\mathbb{C})$ denote the Deligne-Weil group. Let π denote an irreducible admissible representation of $\mathrm{GL}_n(F)$. To π is associated its Lanlgands parameter which is a semisimple n-dimensional representation $\sigma = \sigma(\pi)$ of W'_F via the local Langlands correspondence. We will need only the formalism of the local Langlands correspondence for which we refer the reader to Kudla [5] in the non-Archimedean case and Knapp [4] in the Archimedean case. For the basic properties of representations and epsilon factors of the Deligne-Weil group we refer the reader to Tate [14]. We need the following details about the local Langlands correspondence and local epsilon factors.

(1) If π corresponds to σ as above, then their epsilon factors are equal, i.e., $\varepsilon(s,\pi,\psi) = \varepsilon(s,\sigma,\psi)$. The *root number* is the value of either of these functions at $s = 1/2$; it is a complex number of absolute value 1, and for brevity we denote this as $\varepsilon(\pi,\psi) = \varepsilon(\sigma,\psi)$.

(2) For the additive character ψ of F that we have fixed, let $c(\psi)$ denote the integer such that the largest ideal in the kernel of ψ is $\mathfrak{P}_F^{c(\psi)}$. Let χ be a character of F^*. Let $c(\chi)$ be the conductor of χ which is the least non-negative integer i for which χ is trivial on $1 + \mathfrak{P}_F^i$.
   (a) If χ is unramified, then $\varepsilon(\chi,\psi) = \chi(\varpi_F)^{-c(\psi)}$.
   (b) If χ is ramified, then $\varepsilon(\chi,\psi) = \int_{v_F(x)=c(\psi)-c(\chi)} |x|^{-1/2}\chi^{-1}(x)\psi(x)dx$ where dx is the self-dual measure with respect to ψ.

(3) $\varepsilon(\mathrm{Ind}_{W_E}^{W_F}(\sigma),\psi) = \varepsilon(\sigma,\psi_E)$ for any virtual semisimple representation σ of degree 0 of W_E and $\psi_E = \psi \circ T_{E/F}$.

(4) $\varepsilon(\pi,\psi)\varepsilon(\pi^\vee,\psi) = \omega_\pi(-1)$, where $\pi^\vee$ is the contragredient representation of π.

(5) If π_1 and π_2 are irreducible representations of $\mathrm{GL}_{n_1}(F)$ and $\mathrm{GL}_{n_2}(F)$ then $\pi_1 \times \pi_2$ denotes the irreducible representation of $\mathrm{GL}_{n_1 n_2}(F)$ which is defined by $\sigma(\pi_1 \times \pi_2) = \sigma(\pi_1) \otimes \sigma(\pi_2)$.

(6) If π corresponds to σ then the central character ω_π is the determinant character $\det(\sigma)$ of σ. The conductor of π (see [3]) is the Artin conductor of σ.

(7) If E/F is a quadratic extension and ϕ is a character of $E^\times$ then the determinant of the induced representation $\mathrm{Ind}_{W_E}^{W_F}(\sigma)$ is $\sigma|_{F^*} \cdot \omega_{E/F}$ where $\omega_{E/F}$ is the quadratic character of F associated to E via local class field theory. The Artin conductor of $\mathrm{Ind}_{W_E}^{W_F}(\sigma)$ is given by $e_{E/F}c(\phi) + d(E/F)$ where $e_{E/F}$ is the ramification index of E/F and $d_{E/F}$ is the value of the discriminant of E/F.

As a notational simplification, we suppress the dependence of the root numbers on the additive characters with the understanding that after having fixed ψ on F, for any separable F–algebra L, we take ψ_L to be $\psi \circ T_{L/F}$ where $T_{L/F}$ is the trace map from from L to F.

Lemma 2.1. *Let F be a non-Archimedean local field and let E be a quadratic semisimple F–algebra. let π be an irreducible admissible representation of $\mathrm{GL}_2(F)$ and let ϕ be a character of $E^\times$. Then*

$$\varepsilon(\mathrm{BC}_{E/F}(\pi) \otimes \phi^{-1}) = \varepsilon(\pi \times \pi(\phi^{-1}))\omega_{E/F}(-1).$$

Proof. In [15] (remark on p.1282) this is mentioned for the case in which E is a field. In this case, the lemma may be proved using properties (2), (3) and (4) above. If $E = F \times F$ and $\phi = \phi_1 \otimes \phi_2$ then it is easy to see that both sides are equal to $\varepsilon(\pi \otimes \phi_1^{-1})\varepsilon(\pi \otimes \phi_2^{-1})$. □

Proposition 2.2 (Local reciprocity in the p-adic case)**.** *Let F be a non-Archimedean local field. Let E and K be quadratic semisimple F–algebras. Let ϕ and λ be characters of $E^\times$ and $K^\times$ respectively. Then*

$$\left(\frac{\phi}{\lambda}\right)\left(\frac{\lambda}{\phi}\right) = \phi(-1)\lambda(-1).$$

Proof. By Lemma 2.1 we have

$$\begin{aligned}\left(\frac{\phi}{\lambda}\right) &= \varepsilon(\mathrm{BC}_{K/F}(\pi(\phi)) \otimes \lambda^{-1})\omega_{\pi(\phi)}(-1) \\ &= \varepsilon(\pi(\phi) \times \pi(\lambda^{-1}))\omega_{K/F}(-1)\omega_{E/F}(-1)\lambda(-1).\end{aligned}$$

Therefore we get

$$\left(\frac{\phi}{\lambda}\right)\left(\frac{\lambda}{\phi}\right) = \varepsilon(\pi(\phi) \times \pi(\lambda^{-1}))\varepsilon(\pi(\lambda) \times \pi(\phi^{-1}))\phi(-1)\lambda(-1).$$

Observe that both $\pi(\phi) \times \pi(\lambda^{-1})$ and $\pi(\lambda) \times \pi(\phi^{-1})$ are representations of $\mathrm{GL}_4(F)$ and that one is the contragredient of the other. By properties (4) and (5) above

$$\varepsilon(\pi(\phi) \times \pi(\lambda^{-1}))\varepsilon(\pi(\lambda) \times \pi(\phi^{-1})) = \omega_{\pi(\phi)\times\pi(\lambda^{-1})}(-1) = (\omega^2_{\pi(\phi)}\omega^2_{\pi(\lambda^{-1})})(-1) = 1.$$

This completes the proof of the proposition. □

We now turn to the Archimedean case. Recall that any continuous character of $\mathbb{R}^*$ looks like $(\mathrm{sgn})^\epsilon \otimes |\cdot|_{\mathbb{R}}^t$ where sgn is the sign character, $|\cdot|_{\mathbb{R}}$ is the usual absolute value on $\mathbb{R}$, $\epsilon \in \{0,1\}$ and t is any complex number. Any continuous character of $\mathbb{C}^*$ looks like $\chi_m \otimes |\cdot|_{\mathbb{C}}^t$ where $\chi_m(z) = (z/|z|)^m$ for an integer m, $|\cdot|_{\mathbb{C}}$ is the normalized absolute value on $\mathbb{C}$ and t is any complex number. Fix the character $\psi_{\mathbb{R}}(x) = e^{2\pi i x}$ on $\mathbb{R}^+$ and the character $\psi_{\mathbb{C}}(z) = \psi_{\mathbb{R}}(z+\bar{z})$ on $\mathbb{C}^+$. We need the following well known Archimedean details [4].

Lemma 2.3. *With the above notation we have:*

1. $\varepsilon(|\cdot|_{\mathbb{R}}^t, \psi_{\mathbb{R}}) = 1.$
2. $\varepsilon(\mathrm{sgn} \otimes |\cdot|_{\mathbb{R}}^t, \psi_{\mathbb{R}}) = i.$
3. $\varepsilon(\pi(\chi_m) \otimes |\cdot|_{\mathbb{R}}^t, \psi_{\mathbb{R}}) = i^{m+1}$, *if* $m \neq 0.$
4. $\varepsilon(\chi_m \otimes |\cdot|_{\mathbb{C}}^t, \psi_{\mathbb{C}}) = i^{|m|}.$

Proposition 2.4 (Local reciprocity in the Archimedean case)**.** *Let $F = \mathbb{R}$ or $\mathbb{C}$. Let E and K be quadratic semisimple F–algebras. Let ϕ and λ be characters of $E^\times$ and $K^\times$ respectively. Then we have*

$$\left(\frac{\phi}{\lambda}\right)\left(\frac{\lambda}{\phi}\right) = \phi(-1)\lambda(-1).$$

Proof. We consider three cases. To begin, let $F = \mathbb{R}$ and let $E = K = \mathbb{C}$. Let $\phi = \chi_m \otimes |\cdot|_{\mathbb{C}}^t$ and let $\lambda = \chi_n \otimes |\cdot|_{\mathbb{C}}^s$ with integers m, n and complex numbers s, t. It is easy to see using Lemma 2.3 that

$$\left(\frac{\phi}{\lambda}\right) = i^{|m-n|+|m+n|}(-1)^{m+1}.$$

Hence we have

$$\left(\frac{\phi}{\lambda}\right)\left(\frac{\lambda}{\phi}\right) = (-1)^{m+n} = \phi(-1)\lambda(-1).$$

Next let, $F = \mathbb{R}$, $E = \mathbb{C}$ and $K = \mathbb{R}\times\mathbb{R}$. Let $\phi = \chi_m \otimes |\cdot|_{\mathbb{C}}^t$ and let $\lambda = \lambda_1 \otimes \lambda_2$ with $\lambda_j = (\mathrm{sgn})^{\epsilon_j} \otimes |\cdot|_{\mathbb{R}}^{t_j}$.

Note that

$$\left(\frac{\phi}{\lambda}\right) = \varepsilon(\pi(\phi)\lambda_1^{-1})\varepsilon(\pi(\phi)\lambda_1^{-1})\omega_{\pi(\phi)}(-1).$$

Observe that $\pi(\chi_m \otimes |\cdot|_{\mathbb{C}}^t) \otimes \lambda_j^{-1} = \pi(\chi_m) \otimes |\cdot|_{\mathbb{R}}^{t-t_j}$. Using Lemma 2.3 we get

$$\left(\frac{\phi}{\lambda}\right) = i^{m+1} i^{m+1}(-1)^{m+1} = 1.$$

On the other hand, we have

$$\begin{aligned}\left(\frac{\lambda}{\phi}\right) &= \varepsilon(\mathrm{BC}_{\mathbb{C}/\mathbb{R}}(\pi(\lambda)) \otimes \phi^{-1})\omega_{\pi(\lambda)}(-1)\\ &= \varepsilon(\mathrm{BC}_{\mathbb{C}/\mathbb{R}}(\lambda_1)\phi^{-1})\varepsilon(\mathrm{BC}_{\mathbb{C}/\mathbb{R}}(\lambda_1)\phi^{-1})(\lambda_1\lambda_2)(-1)\\ &= \varepsilon(\chi_{-m} \otimes |\cdot|_{\mathbb{C}}^{t_1-t})\varepsilon(\chi_{-m} \otimes |\cdot|_{\mathbb{C}}^{t_1-t})(\lambda_1\lambda_2)(-1)\\ &= (-1)^m(\lambda_1\lambda_2)(-1) = \phi(-1)\lambda(-1).\end{aligned}$$

Finally, let $F = \mathbb{R}$ or $\mathbb{C}$ and $E = K = F \times F$. Let $\phi = \phi_1 \otimes \phi_2$ be a character of $E^\times$ and let $\lambda = \lambda_1 \otimes \lambda_2$ be a character of $K^\times$. It is easy to see that

$$\left(\frac{\phi}{\lambda}\right) = \varepsilon(\phi_1\lambda_1^{-1})\varepsilon(\phi_2\lambda_1^{-1})\varepsilon(\phi_1\lambda_2^{-1})\varepsilon(\phi_2\lambda_2^{-1})\phi(-1).$$

Using a similar expression by interchanging ϕ and λ we get

$$\left(\frac{\phi}{\lambda}\right)\left(\frac{\lambda}{\phi}\right) = ((\phi_1\lambda_1^{-1})(\phi_2\lambda_1^{-1})(\phi_1\lambda_2^{-1})(\phi_2\lambda_2^{-1}))(-1)\phi(-1)\lambda(-1) = \phi(-1)\lambda(-1).$$

□

3. Applications of the local reciprocity theorem

As a first application of our local reciprocity theorem we now explain the phenomenon of the two examples stated in the introduction. For this we need the following extension of the theorem of Tunnell [15] and Saito [13].

Theorem 3.1. *Let F be a local field of characteristic zero. Let E be a quadratic semisimple F–algebra. Let π be an irreducible admissible infinite dimensional representation of $\mathrm{GL}_2(F)$. Let ϕ be a character of $E^\times$ such that $\phi|_{F^*} = \omega_\pi$. Then ϕ occurs in $\pi|_{E^\times}$ if and only if $\varepsilon(\mathrm{BC}_{E/F}(\pi) \otimes \phi^{-1}, \psi_E)\omega_\pi(-1) = 1$.*

Proof. If F is non-Archimedean and E is not split then this is exactly the theorem of Tunnell and Saito. What is new in this theorem are the other cases.

If F is any local field and $E = F \times F$, then any such ϕ occurs in π. This is a consequence of Kirillov theory for π and is due to Waldspurger [Lemmas 8, 9; [16]]. It suffices to show in this case that $\varepsilon(\mathrm{BC}_{E/F}(\pi) \otimes \phi^{-1}, \psi_E)\omega_\pi(-1) = 1$. Note that $\varepsilon(\mathrm{BC}_{E/F}(\pi) \otimes \phi^{-1}) = \varepsilon(\pi \otimes \phi_1^{-1})\varepsilon(\pi \otimes \phi_2^{-1})$. Since $\phi_1\phi_2 = \omega_\pi$, we get $(\pi \otimes \phi_1^{-1})^\vee = \pi \otimes \phi_2^{-1}$. Hence $\varepsilon(\mathrm{BC}_{E/F}(\pi) \otimes \phi^{-1}) = \omega_{\pi\otimes\phi_1^{-1}}(-1) = \omega_\pi(-1)$.

The only remaining case is when $F = \mathbb{R}$ and $E = \mathbb{C}$. This can be seen using Lemma 2.3 and well known facts about K-types of representations of $\mathrm{GL}_2(\mathbb{R})$ as follows.

Suppose $\pi = \pi(\chi_m)$ is the discrete series representation of lowest weight $m+1$. Let $\phi = \chi_n \otimes |\cdot|_{\mathbb{C}}^t$. If $\phi|_{\mathbb{R}^*}$ is the central character of π then we have $t = 0$ and n and m have opposite parity. We know from the theory of K-types that χ_n occurs in $\pi(\chi_m)$

if and only if $n = \pm(m+1\pm 2r)$ for $r \geq 0$. It is easy to use Lemma 2.3 and see that this very condition on n is equivalent to $\varepsilon(\mathrm{BC}_{E/F}(\pi)\otimes\phi^{-1})\omega_\pi(-1) = 1$.

Suppose π is a principal series representation, by which mean that π is parametrized by a two dimensional representation σ of the Weil group $W_{\mathbb{R}}$ of $\mathbb{R}$ which looks like $\sigma = \sigma_1 \oplus \sigma_2$ where $\sigma_j = (\mathrm{sgn})^{\epsilon_j} \otimes |\cdot|_{\mathbb{R}}^{t_j}$. Let $\phi = \chi_m \otimes |\cdot|_{\mathbb{C}}^{t}$. If ϕ restricts to the central character of π then we have $m \equiv \epsilon_j + \epsilon_j \pmod 2$ and $2t = t_1 + t_2$. One may see using Lemma 2.3 and the condition on the parity of m that $\varepsilon(\mathrm{BC}_{E/F}(\pi)\otimes\phi^{-1})\omega_\pi(-1) = 1$. From the theory of K-types we know that any character of E^* which can possibly occur does indeed occur which follows easily from the $G = BK$ decomposition for $\mathrm{GL}_2(\mathbb{R})$. (Note that this is the Archimedean version of the equivalence of (1) and (3) in Example 1.2.) $\square$

Remark 3.2. For Example 1.1, if χ_n occurs in $\pi(\chi_m)$ then $\left(\frac{\chi_m}{\chi_n}\right) = 1$. Also n and m have opposite parity by central character considerations. Since this gives $\chi_n(-1)\chi_m(-1) = (-1)^{n+m} = -1$ we get that $\left(\frac{\chi_n}{\chi_m}\right) = -1$ which reflects the fact that χ_m does not occur in $\pi(\chi_n)$.

Remark 3.3. For Example 1.2, the first statement is simply the central character criterion. To see that the equivalence of (1) and (2) and also that of (1) and (3) are two facets of the same thing, via our local reciprocity, observe that, $\left(\frac{\lambda}{\phi}\right) = \left(\frac{\lambda'}{\phi}\right)$ since $\mathrm{BC}_{E/F}(\pi(\lambda)) = \pi(\mathrm{BC}_{E/F}(\lambda)) = \pi(\mathrm{BC}_{E/F}(\lambda')) = \mathrm{BC}_{E/F}(\pi(\lambda'))$.

As a second application of the local reciprocity theorem we consider the following question: Let π be a supercuspidal representation of $\mathrm{GL}_2(F)$ such that its Langlands parameter is induced, i.e., $\sigma(\pi) = \mathrm{Ind}_{W_E}^{W_F}(\phi)$. Let K/F be a quadratic extension. Describe all the characters λ of K^* that occurs in π restricted to the non-split torus K^* explicity in terms of the data (E, ϕ, K). For the case $E = K$, using a lemma in [15] along with the local reciprocity theorem, we can give a reasonably complete answer to this question. This question is not new and indeed the results proved in this section can be read off from sections 2, 3 of Tunnell [15] and sections 7, 8 of Dipendra Prasad [11]. The justification for this section is that it gives another point of view to these results and some arguments are seemingly effortless using the local reciprocity theorem. To motivate what one hopes to prove p-adically, we state an elementary lemma involving finite fields. We omit the proof which is an easy calculation using character tables.

Lemma 3.4. *Let $\mathbb{F}_q$ denote the field with q elements. Let $\bar{\pi}(\bar{\phi})$ be the cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_q)$ associated to a regular character $\bar{\phi}$ of $\mathbb{F}_{q^2}^*$. Then*

$$\bar{\pi}(\bar{\phi})|_{\mathbb{F}_{q^2}^*} = \oplus\bar{\lambda}$$

where $\bar{\lambda}$ runs over all characters of $\mathbb{F}_{q^2}^*$ such that $\bar{\lambda}|_{\mathbb{F}_q^*} = \bar{\phi}|_{\mathbb{F}_q^*}$, $\bar{\lambda} \neq \bar{\phi}$ and $\bar{\lambda} \neq \bar{\phi}^s$.

To understand the p-adic analogue of the above result over finite fields we need the following lemma due to Tunnell.

Lemma 3.5 (Tunnell; Lemma 3.1, [15]). *Let π be an irreducible supercuspidal representation of* $\mathrm{GL}_2(F)$ *whose conductor is $c(\pi)$. Let E/F be a separable quadratic extension of F whose ramification index is $e_{E/F}$. Let λ be a character of E^* with conductor $c(\lambda)$ and such that $\lambda|_{F^*} = \omega_\pi$. If $c(\lambda) \geq c(\pi)e_{E/F}/2 - (2 - e_{E/F})$ then λ occurs in π.*

Sketch of Proof. Let π' be the irreducible representation of a quaternion division algebra which corresponds to π via the Jacquet–Langlands correspondence. Since the conductor of λ is *too big*, it cannot occur in π' and hence it occurs in π. ☐

Proposition 3.6. *Let E/F be an unramified quadratic extension with Galois group* $\mathrm{Gal}(E/F) = \{1, s\}$. *Let ϕ be a F-regular character of E^*, i.e., $\phi \neq \phi^s$. Let $\pi(\phi)$ be the corresponding supercuspidal representation of* $\mathrm{GL}_2(F)$. *Let λ be a character of E^* such that $\lambda|_{F^*} = \phi|_{F^*}\omega_{E/F}$. If $c(\lambda) \neq c(\phi)$ then λ occurs in $\pi(\phi)|_{E^*}$.*

Proof. Note that the conductor of $\pi(\phi)$ is given by $c(\pi(\phi)) = 2c(\phi)$. (See property (7) in Section 2.) We consider three cases:

1. $c(\lambda) \geq c(\phi) + 1$. This is exactly the condition on the conductor of λ as in Lemma 3.5 and hence λ occurs in $\pi(\phi)$.
2. $c(\lambda) \leq c(\phi) - 1$ and λ is F-regular. Note that the representation $\pi(\lambda)$ is supercuspidal. Since $c(\phi) \geq c(\lambda) + 1$ by case (1) we get that ϕ occurs in $\pi(\lambda)$. By Theorem 3.1 we get $\left(\frac{\lambda}{\phi}\right) = 1$. By the local reciprocity theorem we get $\left(\frac{\phi}{\lambda}\right) = \lambda(-1)\chi(-1) = \omega_{E/F}(-1) = 1$. Hence λ occurs in $\pi(\phi)$.
3. $c(\lambda) \leq c(\phi) - 1$ and λ is not F-regular. Since λ is Galois invariant, there is a character η of F^* such that $\lambda = \mathrm{BC}_{E/F}(\eta)$. Then $\pi(\lambda)$ is the principal series representation $\pi(\eta, \eta\omega_{E/F})$. Note that $\phi|_{F^*} = \lambda|_{F^*}\omega_{E/F} = \eta^2\omega_{E/F} = \omega_{\pi(\lambda)}$ and hence by Proposition 1.6 of Tunnell [15] we get that $\left(\frac{\lambda}{\phi}\right) = 1$. The rest of the argument is as in Case (2).

☐

Proposition 3.7. *Let E/F be a ramified quadratic extension with Galois group* $\mathrm{Gal}(E/F) = \{1, s\}$. *Let ϕ be a F-regular character of E^*, i.e., $\phi \neq \phi^s$. Let $\pi(\phi)$ be the corresponding supercuspidal representation of* $\mathrm{GL}_2(F)$. *Let λ be a character of E^* such that $\lambda|_{F^*} = \phi|_{F^*}\omega_{E/F}$. Then*

1. *If $c(\lambda) \geq c(\phi) + 1$ then λ occurs in $\pi(\phi)|_{E^*}$.*
2. *If $c(\lambda) \leq c(\phi) - 1$ then λ occurs in $\pi(\phi)|_{E^*}$ if and only if -1 is a square in F.*

Proof. The proof is exactly as that of Proposition 3.6 except, note that $\left(\frac{\phi}{\lambda}\right) = \left(\frac{\lambda}{\phi}\right)\omega_{E/F}(-1)$ and $\omega_{E/F}(-1) = 1$ if and only if -1 is a square in F. ☐

In both the previous two propositions to characterize those λ which occur in $\pi(\phi)$ when λ and ϕ have the same conductors is rather technical (sections 2,3 [15]) and the local reciprocity theorem does not simplify the arguments. However, if $\pi(\phi)$ has depth zero then we can prove the following result.

Proposition 3.8 (Depth zero supercuspidal representations)**.** *Let E/F be an unramified quadratic extension. Let ϕ be an F-regular character of E^* of conductor* 1 *so that $\pi(\phi)$ is a (depth zero) supercuspidal representation. Every depth zero supercuspidal representation of* $\mathrm{GL}_2(F)$ *is of this form. Let λ be a character of E^* of conductor* 1 *such that $\lambda|_{F^*} = \phi|_{F^*}\omega_{E/F}$. Let $\bar{\phi}$ and $\bar{\lambda}$ be the characters of k_E^* induced by ϕ and λ respectively. We have:*

1. *If λ is not F-regular then λ occurs in $\pi(\phi)$.*
2. *If λ is F-regular, then λ does not occur in $\pi(\phi)$ if and only if $\bar{\lambda} = \bar{\phi}$ or $\bar{\phi}^s$ where s is the non-trivial Galois automorphism of E/F.*

Proof. That every depth zero supercuspidal representation is of the form $\pi(\phi)$ for a character ϕ of conductor 1 of a quadratic unramified extension is a well known fact but for lack of a convenient reference we sketch the details here: If π is a depth zero supercuspidal representation then its conductor $c(\pi) = 2$ which follows for instance from [10]. Since the conductor of any supercuspidal representation is at least 2 we get that π is minimal. Since $c(\pi)$ is even we get from [7] that π is, in the terminology of that paper, an unramified supercuspidal representation. From [8] we know that unramified supercuspidal representations are not exceptional and hence the Langlands parameter $\sigma = \sigma(\pi)$ is induced from E, i.e., $\sigma = \mathrm{Ind}_{W_E}^{W_F}(\phi)$. Finally, $2 = c(\pi) = c(\sigma) = 2c(\phi)$ gives that $c(\phi) = 1$.

Suppose λ is not F-regular then arguing as in case (3) of the proof of Proposition 3.6 we can see that λ occurs in $\pi(\phi)$.

If λ is F-regular and $\bar{\lambda} \notin \{\bar{\phi}, \bar{\phi}^s\}$ then using Frobenius reciprocity and Mackey theory [6] we get

$$\begin{aligned}\mathrm{Hom}_{E^*}(\pi(\phi), \lambda) &= \mathrm{Hom}_G(\mathrm{ind}_{Z\cdot\mathrm{GL}_2(\mathcal{O}_F)}^G(\omega \otimes \pi(\bar{\phi})), \mathrm{Ind}_{E^*}^G(\lambda)) \\ &\supset \mathrm{Hom}_{E^*\cap Z\cdot\mathrm{GL}_2(\mathcal{O}_F}(\omega \otimes \pi(\bar{\phi}), \lambda).\end{aligned}$$

where $\omega = \omega_{\pi(\phi)}$ and $\pi(\bar{\phi})$ is the inflation to $\mathrm{GL}_2(\mathcal{O}_F)$ of the cuspidal representation $\bar{\pi}(\bar{\phi})$ of $\mathrm{GL}_2(k_F)$. Now by Lemma 3.4 we have

$$\begin{aligned}\mathrm{Hom}_{E^*\cap Z\cdot\mathrm{GL}_2(\mathcal{O}_F)}(\omega \otimes \pi(\bar{\phi}), \lambda) &= \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_F)}(\pi(\bar{\phi}), \lambda) \\ &= \mathrm{Hom}_{\mathrm{GL}_2(k_F)}(\bar{\pi}(\bar{\phi}), \lambda)) \neq (0).\end{aligned}$$

Finally, if λ is F-regular and $\bar{\lambda} \in \{\bar{\phi}, \bar{\phi}^s\}$ then since $\pi(\phi) = \pi(\phi^s)$ it suffices to consider $\bar{\lambda} = \bar{\phi}$. By Theorem 3.1, λ occurs in $\pi(\phi)$ if and only if $\varepsilon(\pi(\phi)\otimes\lambda^{-1})\omega_{\pi(\phi)} = 1$ if and only if $\varepsilon(\phi\lambda^{-1})\varepsilon(\phi^s\lambda^{-1})\phi(-1) = 1$. Observe that $\phi\lambda^{-1}$ is unramified and since $c(\psi_E) = 0$ we get that $\varepsilon(\phi\lambda^{-1}) = 1$. For brevity let $\chi = \phi^s\lambda^{-1}$, then χ is a character of E^* of conductor 1 and such that $\chi(\varpi_E) = -1$. Define an auxiliary character χ'

such that $\chi = \chi'$ on $\mathcal{O}_E^\times$ and $\chi'(\varpi_E) = 1$. This ensures that χ' is trivial on F^*. Hence by Theorem 3 of Fröhlich–Queyrut [2] we get that $\varepsilon(\chi') = \chi'(\delta)$ for any element δ of E of trace zero. Choose δ to be in $\mathcal{O}_E^\times$. The formula in property (1)(b) of Section 2 gives that $\varepsilon(\chi) = -\varepsilon(\chi')$. Putting all this gives

$$\begin{aligned} \varepsilon(\phi\lambda^{-1})\varepsilon(\phi^s\lambda^{-1})\phi(-1) &= -(\phi^s\lambda^{-1})(\delta)\phi(-1) \\ &= -\phi(-\delta^s)\lambda^{-1}(\delta) = -(\phi\lambda^{-1})(\delta) = -1. \end{aligned}$$

□

As a corollary of the proofs of some parts of the above three propositions it follows that if λ is the base change of a character from F^* then one can explicitly say when λ occurs in π.

Corollary 3.9. *Let E/F be a separable quadratic extension of F. Let ϕ and λ be characters of E^* such that $\lambda|_{F^*} = \phi|_{F^*}\omega_{E/F}$. Then*

1. *If ϕ is the base change of a character of F^* then λ occurs in $\pi(\phi)$.*
2. *If ϕ is F-regular and λ is the base change of a character from F^* then λ occurs in $\pi(\phi)$ if and only if $\omega_{E/F}(-1) = 1$.*

4. Proof of the global reciprocity theorem

For the rest of this paper, we let F denote a number field. Let S_F denote the set of all places of F. We will let S_F^∞ denote the set of all infinite places of F and let $S_F^f = S_F - S_F^\infty$. We let $\mathbb{A}_F$ stand for the ring of adèles of F and $\mathbb{A}_F^\times$ its idèle group.

We let E and K be any quadratic semisimple F–algebras. Let ϕ and λ be regular grössencharacters of E and K respectively, i.e., ϕ is a continuous character of $E^\times\backslash\mathbb{A}_E^\times$ and λ that of $K^\times\backslash\mathbb{A}_K^\times$

We let $\pi(\phi)$ stand for the automorphic induction $\mathrm{AI}_{E/F}(\phi)$. If E is not split then this is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ and if E is split then it is the isobaric sum $\phi_1 \boxplus \phi_2$. A general reference for automorphic induction and the closely related notion of base change is Langlands [9] for GL(2) and Arthur–Clozel [1] for GL(n).

Recall, that the global symbol is defined as

$$\left(\frac{\phi}{\lambda}\right) = \varepsilon(\mathrm{BC}_{K/F}(\pi(\phi)) \otimes \lambda^{-1}).$$

The global proof of the global reciprocity theorem is to observe that

$$\varepsilon(\mathrm{BC}_{K/F}(\pi(\phi)) \otimes \lambda^{-1}) = \varepsilon(\pi(\phi) \times \pi(\lambda^{-1})).$$

In the right hand side, $\pi(\phi) \times \pi(\lambda^{-1})$ denotes the automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$ which is the functorial lift from $\pi(\phi)$ and $\pi(\lambda^{-1})$ which corresponds to the tensor product of a couple of two dimensional representations at the level of dual groups. This lifting is a theorem due to Ramakrishnan [12]. Hence we have

$$\left(\frac{\phi}{\lambda}\right)\left(\frac{\lambda}{\phi}\right) = \varepsilon(\pi(\phi)\times\pi(\lambda^{-1}))\varepsilon(\pi(\lambda)\times\pi(\phi^{-1})) = \omega_{\pi(\phi)\times\pi(\lambda^{-1})}(-1) = 1.$$

The above global proof uses the rather heavy theorem of Ramakrishnan. On the other hand the rather easy local proof of the global reciprocity theorem lies in observing that everything in sight factorizes over the places of F and then to simply use our local reciprocity theorem. The details are as follows.

Let $v \in S_F$. If v splits into two places w_1, w_2 in E then we let $E_v = E_{w_1} \times E_{w_2}$ and put $\phi_v = \phi_{w_1} \otimes \phi_{w_2}$. If w is the only place of E which is above v then we put $E_v = E_w$ and $\phi_v = \phi_w$. Using similar notations for K and λ, we have

$$\left(\frac{\phi}{\lambda}\right) = \left(\frac{\phi}{\lambda}\right)\omega_{\pi(\phi)}(-1) = \prod_{v\in S_F}\left(\frac{\phi_v}{\lambda_v}\right).$$

This infinite product is justified if almost all the factors on the right hand side above are equal to 1. This is stated and proved in the lemma below.

Lemma 4.1. *Let F be a number field. Let E and K be quadratic semisimple F–algebras. Let ϕ and λ be grössencharacters of E and K respectively. With the rest of the notations as above, we have*

$$\left(\frac{\phi_v}{\lambda_v}\right) = 1$$

for almost all $v \in S_F$.

Proof. We will need to pick our various additive characters ψ suitably. Here is how we do this. Take the standard non-trivial additive character $\psi_{\mathbb{Q}}$ on $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$ which is unramified everywhere. For any number field L take $\psi_L = \psi_{\mathbb{Q}} \circ T_{L/\mathbb{Q}}$. For any completion of L take the corresponding local character induced by ψ_L.

Let S_ϕ (resp. S_λ) denote the set of finite places of F which divide the ramified places for ϕ (resp. λ). Let $S_E^{\rm ram}$ (resp. $S_K^{\rm ram}$) denote the finite places of F which divide the absolute different of E (resp. K). For any

$$v \in S_F - (S_\phi \cup S_\lambda \cup S_E^{\rm ram} \cup S_K^{\rm ram} \cup S_F^\infty)$$

it is easy to see that $\left(\frac{\phi_v}{\lambda_v}\right) = 1$ using property (1)(a) of Section 2. $\square$

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