Nonvanishing of certain Rankin-Selberg L-functions *

A. Raghuram

Abstract: *In this article we prove that given a holomorphic cusp form f and any point s_0 in the complex plane, there is a holomorphic cusp form g such that the Rankin-Selberg L-function $L(s, f \times g)$ is non-zero at s_0.*

Résumé: *Dans cet article, on prouve le résultat suivant. Etat donné une forme holomorphe cuspidale f et un point quelquonque du plan complexe, il existe une forme holomorphe cuspidale g telle que la fonction $L(s, f \times g)$ de Rankin-Selberg n'est pas nulle à s_0.*

The aim of this article is to prove that given a holomorphic cusp form f on the upper half plane $\mathfrak{h}$, given any point s_0 in the complex plane and given any positive integer l there is a holomorphic cusp form g of weight $l+1$, which is also an eigenform and a newform and such that the Rankin-Selberg L-function $L(s, f \times g)$ is non-zero at s_0.

One may try to prove such a theorem by averaging. Namely, by choosing a suitable set of 'possible g's' and taking the average of $L(s, f \times g)$ over this set and then isolating a dominant term and showing it is non-zero. In some sense the point of this paper is to say that once such an averaging has been done in one context [6] then some generalities from the theory of automorphic forms takes over and gives our nonvanishing theorem 'almost for free'. The main ingredients in our proof are the notion of base change and automorphic induction for automorphic representations of $GL(2)$ (a general reference for which is [1]) and the main theorem of Rohrlich [6].

After the proof of the main theorem we make various remarks wherein we carefully analyze the choices we make in getting hold of the 'twist' g and in particular say what the level of g can be. For instance, if l is even, then it is possible to arrange the level to be a squarefree product of 2 primes relatively prime to N and one of these primes can be essentially arbitrary. We then point out variations of this theorem wherein

*2000 *Mathematics Subject Classification :* 11F66 (11F67, 11F70, 22E55).

either f and/or g can be a Maass cusp form. Without further ado we state and prove the main theorem.

Theorem. *Let $f \in S_k(\Gamma_0(N), \psi)$, i.e., f is a holomorphic cusp form of weight k, level N, and nebentypus ψ. Let l be any positive integer. Let $s_0 \in \mathbb{C}$. Then there exists $g \in S_{l+1}(\Gamma_0(M), \xi)$ for some level M and some nebentypus ξ such that*

$$L(s_0, f \times g) \neq 0$$

where $L(s, f \times g)$ is the Rankin-Selberg L-function attached to f and g. We can take g to be a Hecke eigenform and also a newform.

Proof : Let $\pi = \pi(f)$ be the cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ associated to f. (See Chapter 5 of [3].) Let K be an imaginary quadratic extension of $\mathbb{Q}$ with discriminant relatively prime to N, the level of f. Let $\Pi = \mathrm{BC}_{K/\mathbb{Q}}(\pi)$ be the base change of π to an automorphic representation Π of $GL_2(\mathbb{A}_K)$. By the assumption on the discriminant of K we have that Π is also cuspidal. (See Lemma 11.3 of [5].) Later on we will be refining our choice of K to have some control over the level of g.

Choose a grossencharacter χ_1 of $\mathbb{A}_K^\times/K^*$ whose infinity component, which is a character of $\mathbb{C}^*$, is given by

$$\chi_{1,\infty}(z) = \left(\frac{z}{|z|}\right)^l$$

and such that χ_1 is unramified at every finite unramified place of K. (It is possible to choose such a character; See Remark 2.) Let S denote a finite set of places of K which contains all the primes dividing Nd_K where d_K is the discriminant of K.

Consider $\Pi \otimes \chi_1$. Let s_1 be any complex number. Apply the main theorem of Rohrlich [6] to $\Pi \otimes \chi_1$ to get a grossencharacter χ_2 which is unramified on S, with trivial infinity component and such that

$$L(s_1, (\Pi \otimes \chi_1) \otimes \chi_2) \neq 0.$$

Let $\chi = \chi_1\chi_2$. By the formalism of base change and automorphic induction we have

$$L(s, \mathrm{BC}_{K/\mathbb{Q}}(\pi) \otimes \chi) = L(s, \pi \times \mathrm{AI}_{K/\mathbb{Q}}(\chi)).$$

Let $\tau = \mathrm{AI}_{K/\mathbb{Q}}(\chi)$ be the automorphic induction of χ. By the choice of $\chi_{1,\infty}$ we have that χ is not Galois invariant (under the Galois group of K over $\mathbb{Q}$) which gives that τ is a cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$.

With the choices made on χ_1 and χ_2 we claim that τ is the cuspidal automorphic representation $\pi(g)$ associated to some holomorphic cusp form g of weight $l + 1$.

Now the infinity component τ_∞ of τ corresponds, via the local Langlands correspondence for $GL_2(\mathbb{R})$, to $\mathrm{Ind}_{\mathbb{C}^*}^{W_\mathbb{R}}(\chi_\infty)$. (See [4].) Here $W_\mathbb{R}$ is the Weil group of $\mathbb{R}$. Since $\chi_\infty = \chi_{1,\infty}$ we get that τ_∞ is the discrete series representation D_{l+1} of $GL_2(\mathbb{R})$. (See equations (2.1b) and (3.4) of [4]. As a word of warning, our D_{l+1} is D_l in [4]. This is done to emphasize that a subscript n in the notation for the discrete series D_n should be the weight n of the modular form it comes from and is characterized in the representation by $SO(2)$ acting on this form via the character $e^{i\theta} \mapsto e^{-in\theta}$.)

Let ω_τ denote the central character of τ. The infinity component of ω_τ, denoted $\omega_{\tau,\infty}$ is equal to ω_{τ_∞} the central character of τ_∞. Now ω_{τ_∞} is, via the local Langlands correspondence, the determinant of $\mathrm{Ind}_{\mathbb{C}^*}^{W_\mathbb{R}}(\chi_\infty)$. A pleasant exercise (whose details we omit) gives that $\omega_{\tau,\infty}$ is $(\mathrm{sgn})^{l+1}$ and so it is trivial on $\mathbb{R}_{>0}$.

Let $\mathfrak{f}_\tau$ be the conductor of τ and put $M = |\mathfrak{f}_\tau|$ and $\xi = \omega_\tau$. Apply Theorem 5.19 of [3] to get a $g \in S_{l+1}(\Gamma_0(M), \xi)$ such that $\tau = \pi(g)$. Note that there is an eigenform which is also a newform satisfying this. Hence what we have proved till now is that

$$L(s_1, \pi \times \tau) = L(s_1, \pi(f) \times \pi(g)) \neq 0.$$

Note that $L(s, \pi(f) \times \pi(g)) = L(s + t, f \times g)$ where t depends only on l and k. Now taking $s_1 = s_0 - t$ finishes the proof. □

Remark 1 (Choice of K, χ_1, and χ_2) Let p be a prime not dividing N such that $p \equiv 3 \pmod 4$ and let $K = \mathbb{Q}(\sqrt{-p})$. So we have that p is the only prime which ramifies in K and let $p\mathcal{O}_K = \mathfrak{p}^2$. (Here $\mathcal{O}_K$ is the ring of integers of K.)

We show now that it is possible to choose a unitary character χ_1 of $\mathbb{A}_K^\times/K^*$ whose infinity component $\chi_{1,\infty}$ is the character which sends $z \in \mathbb{C}^*$ to $(z/|z|)^l$ and such that χ_1 is unramified at all finite places except possibly at $\mathfrak{p}$. (For any finite place v of K, we let K_v denote the completion of K at v and let U_v denote the corresponding group of units, and let U_v^1 denote the first filtration subgroup of U_v.)

Now $K^* \cap (K_\infty^* U_\mathfrak{p}^1 \prod_{v \neq \infty, \mathfrak{p}} U_v) = \{1\}$ and so we can extend $z \mapsto (z/|z|)^l$ to a character of $K^*(K_\infty^* U_\mathfrak{p}^1 \prod_{v \neq \infty, \mathfrak{p}} U_v)/K^*$ by making it trivial on all the finite factors. Also since $K^* \cap (K_\infty^* \prod_{v \neq \infty} U_v) = \{\pm 1\}$ we can further extend up to $K^*(K_\infty^* \prod_{v \neq \infty} U_v)/K^*$ by asking that the local character at $\mathfrak{p}$ take the value $(-1)^l$ on -1. By standard character theory of locally compact abelian groups this character can be extended to a character of $\mathbb{A}_K^\times/K^*$ since this group contains $K^*(K_\infty^* \prod_{v \neq \infty} U_v)/K^*$ as subgroup of finite index. *Note that the conductor of χ_1 is $\mathfrak{p}$ if l is odd and χ_1 is unramified at all finite places if l is even.*

In the proof of the main theorem, χ_2 was a unitary character of K coming out of Rohrlich [6]. It is possible to choose it a little carefully. The constraints on the conductor $\mathfrak{q}$ of χ_2 are dictated by Proposition 2 of [6]. (See also p.394 of [6].) These are that the absolute norm $\mathbb{N}(\mathfrak{q})$ of $\mathfrak{q}$ should be a product of distinct primes, that $\mathbb{N}(\mathfrak{q}) \gg 0$ and the wide ray class number modulo $\mathfrak{q}$, denoted $h^*(\mathfrak{q})$, should satisfy $h^*(\mathfrak{q}) > \mathbb{N}(\mathfrak{q})^{1-\epsilon}$ for every $\epsilon > 0$. With this in mind, let $\mathfrak{q}$ be a prime in K such that $\mathfrak{q} \cap \mathbb{Q} = (q)$ splits in K and $\mathfrak{q}$ is not in S. It is an easy exercise to see that the wide ray class number satisfies $h^*(\mathfrak{q}) \geq \mathbb{N}(\mathfrak{q})/2$. Hence wide ray class characters of K modulo $\mathfrak{q}$ satisfy Rohrlich's requirement (as long as $\mathbb{N}(\mathfrak{q}) = q \gg 0$). *So χ_2 is a wide primitive ray class character of conductor a prime $\mathfrak{q}$ as above.*

Remark 2 (The level and nebentypus of g) The level M of g is the absolute value of $\mathfrak{f}_\tau$ the conductor of $\tau = \mathrm{AI}_{K/\mathbb{Q}}(\chi)$. The conductor of τ is the product of all its local conductors. We analyze this case by case. *At the prime p:* Here we need two further cases. *If l is even* then $\chi_\mathfrak{p}$ is unramified and the Langlands parameter corresponding to τ_p is $\mathrm{Ind}_{W_{K_\mathfrak{p}}}^{W_\mathbb{Q}}(\chi_\mathfrak{p})$. Using Proposition 4(b) on p.158 of [2] (which computes the local Artin conductor of an induced representation) gives that the conductor of τ_p is p. *If l is odd* then $\chi_\mathfrak{p}$ has level 1 and this same proposition gives that the conductor of τ_p is p^2. *At the prime q:* We have $q\mathcal{O}_K = \mathfrak{q}\mathfrak{q}'$ and $\chi_{\mathfrak{q}'}$ is unramified whereas $\chi_\mathfrak{q}$ has level 1. Hence τ_q is an irreducible principal series representation whose Langlands parameter is $\chi_\mathfrak{q} \oplus \chi_{\mathfrak{q}'}$ and this has conductor q. *At any prime other than p or q:* It is clear at that all these local representations are unramified.

To summarize what we have is that *the level M of g is p^2q if l is odd and is pq if l is even, where p and q are two primes such that $(N, pq) = 1$, $p \equiv 3 \pmod 4$, q splits in $K = \mathbb{Q}(\sqrt{-p})$ and also $q \gg 0$.*

Calculating the nebentypus ξ of g is a classical computation of Hecke and in 'modern' language it is the central character of τ and this is the determinant of the induced representation. For our set up this gives ξ, as a character of $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^*$, to be $\xi = \omega_{K/\mathbb{Q}}\chi$ where $\omega_{K/\mathbb{Q}}$ is the character associated to the quadratic extension $K/\mathbb{Q}$ and the χ on the right hand side is the restriction of χ to $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^*$.

Remark 3 (A couple of variants of the theorem) One variant of the theorem is as follows. We can begin with f being a Maass cusp form, i.e., f is a bounded, $\Gamma_0(N)$ invariant, real analytic function on the upper half plane $\mathfrak{h}$ and such that f is an eigenfunction for the Laplace-Beltrami operator on $\mathfrak{h}$. (This eigenfunction condition looks like $\Delta(f) = \frac{1-s^2}{4}f$ where s is a purely imaginary number or a real number with absolute value less than 1.) Now let $\pi = \pi(f)$ be the cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ associated to f. The rest of the proof goes through mutatis mutandis to get hold of a holomorphic cusp form g with very similar kind of restrictions, such that $L(s_0, f \times g) \neq 0$.

Alternatively we begin with f being either a holomorphic cusp form or a Maass cusp form and then considering the base change of $\pi = \pi(f)$ to K, but now taking K to be a real quadratic extension. Since K is real quadratic the infinity part of $\mathbb{A}_K^\times$ is $\mathbb{R}^* \times \mathbb{R}^*$. So we can choose a unitary character χ_1 of $\mathbb{A}_K^\times/K^*$ such that $\mathrm{AI}_{K/\mathbb{Q}}(\chi_1)$ has a unitary principal series representation of $GL_2(\mathbb{R})$ with central character trivial on $\mathbb{R}_{>0}$. Now going through the entire proof with this π and χ_1 we can get hold of a Maass cusp form g such that $L(s_0, f \times g) \neq 0$. (These are the Maass cusp forms as constructed in Section 7C of [3] but with a 'Rohrlich twist'.)

Acknowledgements. I thank Ram Murty who posed this question to me in an informal conversation. I thank Eknath Ghate, Kumar Murty and Joe Repka for some helpful comments.

References

[1] J. Arthur and L. Clozel, *Simple Algebras, Base Change, and the Advanced theory of the trace formula,* Annals of Mathematics Studies, No. 120, Princeton University Press, (1990).

[2] J.W.S. Cassels and A. Frohlich, *Algebraic Number Theory*, Academic Press, 1967.

[3] S. Gelbart, *Automorphic forms on adele groups,* Annals of Mathematics Studies, No. 83, Princeton University Press, (1975).

[4] A. Knapp, *Local Langlands correspondence : The Archimedean case,* Proc. Symp. of Pure Math., Vol 55, part II, 393-410(1994).

[5] R. Langlands, *Base change for* $GL(2)$, Annals of Mathematics Studies, No. 96, Princeton University Press, (1980).

[6] D. Rohrlich, *Nonvanishing of L-functions for* $GL(2)$, Invent. Math., 97, 381-403(1989).

Address : Department of Mathematics, University of Toronto, 100 St. George Street, Toronto, ON, CANADA, M5S 3G3; raghuram@math.toronto.edu Current Address : School of Mathematics, Tata Institute of Fundamental Research, Dr. Homi Bhabha Road, Colaba, Mumbai-400005, INDIA; raghuram@math.tifr.res.in