FUNCTORIALITY AND SPECIAL VALUES OF L-FUNCTIONS

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Abstract. This is a semi-expository article concerning Langlands functoriality and Deligne's conjecture on the special values of L-functions. The emphasis is on symmetric power L-functions associated to a holomorphic cusp form.

Contents

1. Introduction

Langlands functoriality principle reduces the study of automorphic forms on the adèlic points of a reductive algebraic group to those of an appropriate general linear group. In particular, every automorphic L-function on an arbitrary reductive group must be one for a suitable GL_n. One should therefore be able to reduce the study of of special values of an automorphic L-function to those of a principal L-function of Godement and Jacquet on GL_n.

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While the integral representations of Godement and Jacquet do not seem to admit a cohomological interpretation, there is a recent work of J. Mahnkopf [26] [27] which provides us with such an interpretation for certain Rankin–Selberg type integrals. In particular, modulo a nonvanishing assumption on local archimedean Rankin–Selberg product L-functions for forms on $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$, he defines a pair of periods, which seem to be in accordance with those of Deligne [9] and Shimura [42]. This work of Mahnkopf is quite remarkable and requires the use of both Rankin–Selberg and Langlands–Shahidi methods in studying the analytic (and arithmetic) properties of L-functions. His work therefore brings in the theory of Eisenstein series to play an important role. In §6 we briefly review this work of Mahnkopf.

This article is an attempt to test the philosophy–to study the special values of L-functions while using functoriality–by means of recent cases of functoriality established for symmetric powers of automorphic forms on GL_2 [17] [21]. While a proof of the precise formulae in the conjectures of Deligne [9] still seem to be out of reach, we expect to be able to prove explicit connections between the special values of symmetric power L-functions twisted by Dirichlet characters and those of the original symmetric power L-functions using this work of Mahnkopf. These relations are formulated in this paper as Conjecture 7.1 which seems to be compatible with the more general conjectures of Blasius [3] and Panchiskin [32].

A standard assumption made in the study of special values of L-functions is that the representations (to which are attached the L-functions) are cohomological. This is the case in Mahnkopf's work. A global representation being cohomological is entirely determined by the archimedean components. For representations which are symmetric power lifts of a cusp form on GL_2 we have the following fact. Consider a holomorphic cusp form on the upper half plane of weight k. This corresponds to a cuspidal automorphic representation, which is cohomological if $k \geq 2$, and any symmetric power lift, if cuspidal, is also cohomological. (If the weight $k = 1$ then the representation is not cohomological, and furthermore none of the symmetric power L-functions have any critical points.) In §5 we review representations with cohomology in the case of GL_n.

We recall the functorial formalism for symmetric powers in §2. We then review Deligne's conjecture for the special values of symmetric power L-functions in §3 and give a brief survey as to which cases are known so far. In §4 we sketch a proof of the conjecture for dihedral representations; the details will appear elsewhere [33].

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2. Symmetric powers and functoriality

In this section we recall the formalism of Langlands functoriality especially for symmetric powers. We will be brief here as there are several very good expositions of the principle of functoriality; see for instance [8, Chapter 2].

Let F be a number field and let $\mathbb{A}_F$ be its adèle ring. We let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, by which we mean that, for some $s \in \mathbb{R}$, $\pi \otimes |\cdot|^s$ is an irreducible summand of

$$L^2_{\mathrm{cusp}}(\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F), \omega)$$

the space of square-integrable cusp forms with central character ω. We have the decomposition $\pi = \otimes'_v \pi_v$ where v runs over all places of F and π_v is an irreducible admissible representation of $\mathrm{GL}_2(F_v)$.

The local Langlands correspondence for GL_2 (see [24] and [23] for the p-adic case and [22] for the archimedean case), says that to π_v is associated a representation $\sigma(\pi_v) : W'_{F_v} \to \mathrm{GL}_2(\mathbb{C})$ of the Weil–Deligne group W'_{F_v} of F_v. (If v is infinite, we take $W'_{F_v} = W_{F_v}$.) Let $n \geq 1$ be an integer. Consider the n-th symmetric power of $\sigma(\pi_v)$ which is an $n+1$ dimensional representation. This is simply the composition of $\sigma(\pi_v)$ with $\mathrm{Sym}^n : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{n+1}(\mathbb{C})$. Appealing to the local Langlands correspondence for GL_{n+1} ([14], [15], [22], [23]) we get an irreducible admissible representation of $\mathrm{GL}_{n+1}(F_v)$ which we denote as $\mathrm{Sym}^n(\pi_v)$. Now define a global representation of $\mathrm{Sym}^n(\pi)$ of $\mathrm{GL}_{n+1}(\mathbb{A}_F)$ by

$$\mathrm{Sym}^n(\pi) := \otimes'_v \mathrm{Sym}^n(\pi_v).$$

Langlands principle of functoriality predicts that $\mathrm{Sym}^n(\pi)$ is an automorphic representation of $\mathrm{GL}_{n+1}(\mathbb{A}_F)$, i.e., it is isomorphic to an irreducible subquotient of the representation of $\mathrm{GL}_{n+1}(\mathbb{A}_F)$ on the space of automorphic forms [4, §4.6]. If ω_π is the central character of π then $\omega_\pi^{n(n+1)}$ is the central character of $\mathrm{Sym}^n(\pi)$. Actually it is expected to be an isobaric automorphic representation. (See [8, Definition 1.1.2] for a definition of an isobaric representation.) The principle of functoriality for the n-th symmetric power is known for $n = 2$ by Gelbart–Jacquet [11]; for $n = 3$ by Kim–Shahidi [21]; and for $n = 4$ by Kim [17]. For certain special forms π, for instance, if π is dihedral, tetrahedral, octahedral or icosahedral, it is know for all n (see [18] [33]).

3. Deligne's conjecture for symmetric power L-functions

Deligne's conjecture on the special values of L-functions is a conjecture which predicts the *transcendental* parts of the special values of motivic L-functions at critical points. The definitive reference is Deligne's article [9]. We begin by introducing the symmetric power L-functions, which are examples of motivic L-functions, and then state Deligne's conjecture for these L-functions.

3.1. **Symmetric power L-functions.** Let $\varphi \in S_k(N, \omega)$, i.e., φ is a holomorphic cusp form on the upper half plane, for $\Gamma_0(N)$, of weight k, and nebentypus character ω. Let $\varphi(z) = \sum_{n=1}^{\infty} a_n q^n$ be the Fourier expansion of φ at infinity. We let $L(s, \varphi)$ stand for the completed L-function associated to φ and let $L_f(s, \varphi)$ stand for its finite part. Assume that φ is a primitive form in $S_k(N, \omega)$. By primitive, we mean that it is an eigenform, a newform and is normalized such that $a_1(\varphi) = 1$. In a suitable right half plane the finite part $L_f(s, \varphi)$ is a Dirichlet series with an Euler product

$$L_f(s, \varphi) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_p L_p(s, \varphi),$$

where, for all primes p, we have

$$L_p(s, \varphi) = (1 - a_p p^{-s} + \omega(p) p^{k-1-2s})^{-1} = (1 - \alpha_{p,\varphi} p^{-s})^{-1} (1 - \beta_{p,\varphi} p^{-s})^{-1},$$

with the convention that if $p|N$ then $\beta_{p,\varphi} = 0$. We let $\mathrm{Supp}(N)$ stand for the set of primes dividing N and let $S = \mathrm{Supp}(N) \cup \{\infty\}$.

For any $n \geq 1$, the partial n-th symmetric power L-function is defined as

$$L^S(s, \mathrm{Sym}^n \varphi) = \prod_{p \notin S} L_p(s, \mathrm{Sym}^n \varphi),$$

where, for all $p \notin S$, we have

$$L_p(s, \mathrm{Sym}^n \varphi) = \prod_{i=0}^{n} (1 - \alpha_{p,\varphi}^i \beta_{p,\varphi}^{n-i} p^{-s})^{-1}.$$

Using the local Langlands correspondence the partial L-function can be completed by defining local factors $L_p(s, \mathrm{Sym}^n \varphi)$ for $p \in S$ and the completed L-function, which is a product over all p including ∞, will be denoted as $L(s, \mathrm{Sym}^n \varphi)$. The Langlands program predicts that $L(s, \mathrm{Sym}^n \varphi)$, which is initially defined only in a half plane, admits a meromorphic continuation to the entire complex plane and that it has all the usual properties an automorphic L-function is supposed to have. This is known for $n \leq 4$ from the works of several people including Hecke, Shimura, Gelbart–Jacquet, Kim and Shahidi. It is also known for all n for cusp forms of a special type, for instance, if the representation corresponding to the cusp form is dihedral or the other polyhedral types. (The reader is referred to the same references as in the last paragraph of the previous section.)

3.2. **Deligne's conjecture.** Let φ be a primitive form in $S_k(N, \omega)$. Let $M(\varphi)$ be the motive associated to φ. This is a rank two motive over $\mathbb{Q}$ with coefficients in the field $\mathbb{Q}(\varphi)$ generated by the Fourier coefficients of φ. (We refer the reader to Deligne [9] and Scholl [38] for details about $M(\varphi)$.) The L-function $L(s, M(\varphi))$ associated to this motive is $L(s, \varphi)$. Given the motive $M(\varphi)$ there are nonzero complex numbers, called Deligne's periods, $c^{\pm}(M(\varphi))$ associated to it. Similarly, for the symmetric powers $\mathrm{Sym}^n(M(\varphi))$, we have the periods $c^{\pm}(\mathrm{Sym}^n(M(\varphi))$. In [9, Proposition 7.7] the periods for the symmetric powers are related to the periods of $M(\varphi)$. The explicit formulae therein have a quantity $\delta(M(\varphi))$ which is essentially the Gauss sum of the nebentypus character ω and is given by

$$\delta(M(\varphi)) \sim (2\pi i)^{1-k}\mathfrak{g}(\omega) := (2\pi i)^{1-k}\sum_{u=0}^{c-1}\omega_0(u)\exp(-2\pi i u/c),$$

where c is the conductor of ω and ω_0 is the primitive character associated to ω. We will denote the right hand side by $\delta(\omega)$. For brevity, we will denote $c^{\pm}(M(\varphi))$ by $c^{\pm}(\varphi)$. Recall [9, Definition 1.3] that an integer m is *critical* for any motivic L-function $L(s, M)$ if both $L_\infty(s, M)$ and $L_\infty(1-s, M^\vee)$ are regular at $s = m$. We now state Deligne's conjecture [9, Section 7] on the special values of the symmetric power L-functions.

Conjecture 3.1. *Let φ be a primitive form in $S_k(N, \omega)$. There exist nonzero complex numbers $c^{\pm}(\varphi)$ such that*

(1) *If m is a critical integer for $L_f(s, \mathrm{Sym}^{2l+1}\varphi)$, then*
$$L_f(m, \mathrm{Sym}^{2l+1}\varphi) \sim (2\pi i)^{m(l+1)}\, c^{\pm}(\varphi)^{(l+1)(l+2)/2}\, c^{\mp}(\varphi)^{l(l+1)/2}\, \delta(\omega)^{l(l+1)/2},$$
where $\pm = (-1)^m$.

(2) *If m is a critical integer for $L_f(s, \mathrm{Sym}^{2l}\varphi)$, then*
$$L_f(m, \mathrm{Sym}^{2l}\varphi) \sim \begin{cases} (2\pi i)^{m(l+1)}\, (c^+(\varphi)c^-(\varphi))^{l(l+1)/2}\, \delta(\omega)^{l(l+1)/2} & \text{if } m \text{ is even,} \\ (2\pi i)^{ml}\, (c^+(\varphi)c^-(\varphi))^{l(l+1)/2}\, \delta(\omega)^{l(l-1)/2} & \text{if } m \text{ is odd.} \end{cases}$$

By $\sim$ we mean up to an element of $\mathbb{Q}(\varphi)$.

It adds some clarity to write down explicitly the statement of the conjecture for the n-th symmetric power, in the special cases $n = 1, 2, 3, 4$, and while doing so we also discuss about how much is known in these cases.

Let m be a critical integer for $L_f(s, \varphi)$. Then Conjecture 3.1 takes the form

$$L_f(m, \varphi) \sim (2\pi i)^m c^{\pm}(\varphi), \tag{3.2}$$

where $\pm = (-1)^m$. In this context, the conjecture is known and is a theorem of Shimura [41] [42]. Shimura relates the required special values to quotients of certain Petersson inner products, whose rationality properties can be studied.

Let m be a critical integer for $L_f(s, \mathrm{Sym}^2\varphi)$. Then Conjecture 3.1 takes the form

$$L_f(m, \mathrm{Sym}^2\varphi) \sim \begin{cases} (2\pi i)^{2m}(c^+(\varphi)c^-(\varphi))\delta(\omega) & \text{if m is even,} \\ (2\pi i)^m(c^+(\varphi)c^-(\varphi)) & \text{if m is odd.} \end{cases} \tag{3.3}$$

The conjecture is known in this case and is due to Sturm [43] [44]. Sturm uses an integral representation for the symmetric square L-function due to Shimura [40].

Let m be a critical integer for $L_f(s, \mathrm{Sym}^3\varphi)$. Then Conjecture 3.1 takes the form

$$L_f(m, \mathrm{Sym}^3\varphi) \sim (2\pi i)^{2m} c^{\pm}(\varphi)^3 c^{\mp}(\varphi)\delta(\omega), \tag{3.4}$$

where $\pm = (-1)^m$. The conjecture is known in this case and is due to Garrett and Harris [10]. The main thrust of that paper is to prove a theorem on the special values of certain triple product L-functions $L(s, \varphi_1 \times \varphi_2 \times \varphi_3)$. Deligne's conjecture for motivic L-functions predicts the special values of such triple product L-functions, for which an excellent reference is Blasius [2]. Via a standard argument, the case $\varphi_1 = \varphi_2 = \varphi_3 = \varphi$, gives the special values of the symmetric cube L-function for φ. This was reproved by Kim and Shahidi [19] emphasizing finiteness of these L-values which follows from their earlier work [20].

Let m be a critical integer for $L_f(s, \mathrm{Sym}^4\varphi)$. Then Conjecture 3.1 takes the form

$$L_f(m, \mathrm{Sym}^4\varphi) \sim \begin{cases} (2\pi i)^{3m}(c^+(\varphi)c^-(\varphi))^3\delta(\omega)^3 & \text{if m is even,} \\ (2\pi i)^{2m}(c^+(\varphi)c^-(\varphi))^3\delta(\omega) & \text{if m is odd.} \end{cases} \tag{3.5}$$

In general the conjecture is not known for higher ($n \geq 4$) symmetric power L-functions. Although, if φ is dihedral, then we have verified the conjecture for any symmetric power; see §4.

We remark that a prelude to this conjecture was certain calculations made by Zagier [46] wherein he showed that such a statement holds for the n-th symmetric power L-function, with $n \leq 4$, of the Ramanujan Δ-function.

3.3. **Critical points.** As recalled above, an integer m is *critical* for any motivic L-function $L(s, M)$ if both $L_\infty(s, M)$ and $L_\infty(1 - s, M^\vee)$ are regular at $s = m$. For example, if $M = \mathbb{Z}(0) = H^*(\text{Point})$ be the trivial motive, then $L(s, M)$ is the Riemann zeta function $\zeta(s)$ [9, §3.2]. Then $L_\infty(s, M) = \pi^{-s/2}\Gamma(s/2)$. It is an easy exercise to see that an integer m is critical for $\zeta(s)$ if m is an even positive integer or an odd negative integer. More generally, as in Blasius [2], one can calculate the critical points for any motivic L-function in terms of the Hodge numbers of the corresponding motive. For the specific L-functions at hand, namely the symmetric power L-functions, one explicitly knows the L-factors at infinity [28] using which it is a straightforward exercise to calculate the critical points. In the following two lemmas we record the critical points of the n-th symmetric power L-function associated to a modular form φ. (For more details see [33].)

Lemma 3.6. *Let φ be a primitive cusp form of weight k. The set of critical integers for $L_f(s, \mathrm{Sym}^{2r+1}\varphi)$ is given by integers m with*

$$r(k-1) + 1 \leq m \leq (r+1)(k-1).$$

Lemma 3.7. *Let φ be a primitive cusp form of weight k. The set of critical integers for $L_f(s, \mathrm{Sym}^{2r}\varphi)$ is given below.*

(1) *If r is odd and k is even, then*

$$\{(r-1)(k-1)+1, (r-1)(k-1)+3, \ldots, r(k-1);\ r(k-1)+1, r(k-1)+3, \ldots, (r+1)(k-1)\}.$$

(2) *If r and k are both odd, then*

$$\{(r-1)(k-1)+1, (r-1)(k-1)+3, \ldots, r(k-1)-1;\ r(k-1)+2, r(k-1)+4, \ldots, (r+1)(k-1)\}.$$

(3) *If r and k are both even, then*

$$\{(r-1)(k-1)+2, (r-1)(k-1)+4, \ldots, r(k-1)-1;\ r(k-1)+2, r(k-1)+4, \ldots, (r+1)(k-1)-1\}.$$

(4) *If r is even and k is odd, then*

$$\{(r-1)(k-1)+1, (r-1)(k-1)+3, \ldots, r(k-1)-1;\ r(k-1)+2, r(k-1)+4, \ldots, (r+1)(k-1)\}.$$

Remark 3.8. Here are some easy observations based on the above lemmas.

(1) If $k = 1$ then $L_f(s, \mathrm{Sym}^n\varphi)$ does not have any critical points for any $n \geq 1$. In particular, this is the case if φ is a cusp form which is tetrahedral, octahedral or icosahedral [33].
(2) If $k = 2$ then $L_f(s, \mathrm{Sym}^n\varphi)$ has a critical point if and only if n is not a multiple of 4; further $L_f(s, \mathrm{Sym}^{2r+1}\varphi)$ has exactly one critical point $m = r+1$; and if r is odd $L_f(s, \mathrm{Sym}^{2r}\varphi)$ has two critical points $r, r+1$. This applies in particular for symmetric power L-functions of elliptic curves.
(3) Let m be a critical integer for $L_f(s, \mathrm{Sym}^{2r}\varphi)$. Then m is even if and only if m is to the right of the center of symmetry.

4. Dihedral calculations

A primitive form φ is said to be *dihedral* if the associated cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, denoted $\pi(\varphi)$, is the automorphic induction of an idèle class character, say χ, of a quadratic extension $K/\mathbb{Q}$. This is denoted as $\pi(\varphi) = \mathrm{AI}_{K/\mathbb{Q}}(\chi)$. (Since φ is a holomorphic modular form, in this situation, K is necessarily an imaginary quadratic extension.) In [33] we have proved Deligne's conjecture for the special values of any symmetric power L-function for such a dihedral form. In this section we summarize the main results of those calculations while referring the reader to [33] for all the proofs.

Recall from Remark 3.8 that if the weight $k = 1$ then there are no critical integers for $L_f(s, \mathrm{Sym}^n\varphi)$. It is easy to see [33] that if $\pi(\varphi) = \mathrm{AI}_{K/\mathbb{Q}}(\chi)$ and some nonzero power of χ is Galois invariant (under the Galois group of $K/\mathbb{Q}$) then $k = 1$. Hence we may, and henceforth shall, assume that for every nonzero integer n, χ^n is not Galois invariant. The following lemma describes the isobaric decomposition of a symmetric power lifting of a dihedral cusp form.

Lemma 4.1. *Let χ be an idèle class character of an imaginary quadratic extension $K/\mathbb{Q}$; assume that χ^n is not Galois invariant for any nonzero integer n. Let $\chi_{\mathbb{Q}}$ denote the restriction of χ to the idèles of $\mathbb{Q}$. Then we have*

$$\begin{aligned}\mathrm{Sym}^{2r}(\mathrm{AI}_{K/\mathbb{Q}}(\chi)) &= \boxplus_{a=0}^{r-1}\mathrm{AI}_{K/\mathbb{Q}}(\chi^{2r-a}\chi'^{a}) \boxplus \chi_{\mathbb{Q}}^{r},\\ \mathrm{Sym}^{2r+1}(\mathrm{AI}_{K/\mathbb{Q}}(\chi)) &= \boxplus_{a=0}^{r}\mathrm{AI}_{K/\mathbb{Q}}(\chi^{2r+1-a}\chi'^{a}),\end{aligned}$$

where χ' is the nontrivial Galois conjugate of χ.

Note that every isobaric summand above is either cuspidal or is one dimensional. This lemma can be recast in terms of L-functions. For an idèle class character χ of an imaginary quadratic extension $K/\mathbb{Q}$, we let φ_χ denote the primitive cusp form such that $\pi(\varphi_\chi) = \mathrm{AI}_{K/\mathbb{Q}}(\chi)$. If $\varphi_\chi \in S_k(N,\omega)$ then $\omega\omega_K = \chi_{\mathbb{Q}}$, where we make the obvious identification of classical Dirichlet characters and idèle class characters of $\mathbb{Q}$, and ω_K denotes the quadratic idèle class character of $\mathbb{Q}$ associated to K via global class field theory.

Lemma 4.2. *The symmetric power L-functions of φ_χ decompose as follows:*

$$\begin{aligned} L_f(s, \mathrm{Sym}^{2r}\varphi_\chi) &= L_f(s - r(k-1), (\omega\omega_K)^r) \prod_{a=0}^{r-1} L_f(s - a(k-1), \varphi_{\chi^{2(r-a)}}, \omega^a) \\ &= L_f(s - r(k-1), (\omega\omega_K)^r) \prod_{a=0}^{r-1} L_f(s - a(k-1), \varphi_{\chi^{2(r-a)}}, (\omega\omega_K)^a). \\ L_f(s, \mathrm{Sym}^{2r+1}\varphi_\chi) &= \prod_{a=0}^{r} L_f(s - a(k-1), \varphi_{\chi^{2(r-a)+1}}, \omega^a) \\ &= \prod_{a=0}^{r} L_f(s - a(k-1), \varphi_{\chi^{2(r-a)+1}}, (\omega\omega_K)^a). \end{aligned}$$

We can now use the results of Shimura [41] [42] and classical theorems on special values of abelian (degree 1) L-functions for the factors on the right hand side of the above decompositions to prove Deligne's conjecture on the special values of $L_f(s, \mathrm{Sym}^n\varphi_\chi)$. The proof is an extended exercise in keeping track of various constants after one has related the periods of the cusp form φ_{χ^n} to the periods of φ_χ. We state this as the following theorem.

Theorem 4.3 (Period relations for dihedral forms)**.** *For any positive integer n we have the following relations:*

(1) $c^+(\varphi_{\chi^n}) \sim c^+(\varphi_\chi)^n$,
(2) $c^-(\varphi_{\chi^n}) \sim c^+(\varphi_\chi)^n \mathfrak{g}(\omega_K)$,

where $\sim$ means up to an element of $\mathbb{Q}(\chi)$–the field generated by the values of χ, and $\mathfrak{g}(\omega_K)$ is the Gauss sum of ω_K.

5. Representations with cohomology

In the study of special values of L-functions, if the L-function at hand is associated to a cuspidal automorphic representation, then a standard assumption made on the representation is that it contributes to cuspidal cohomology. This cohomology space admits a rational structure and the periods, which give the transcendental parts of the special values, come by comparing this rational structure to the rational structure on the Whittaker model of the representation at hand. This approach to the study of special values is originally due to Harder [12] and since then pursued by several authors and in particular by Mahnkopf [27].

The purpose of this section, after setting up the context, is to record Theorem 5.5 which says that the n-th symmetric power lift of a cohomological cusp form on GL_2, if cuspidal, contributes to cuspidal cohomology of GL_{n+1}. This theorem is essentially due to Labesse and Schwermer [25]. We then digress a little and discuss the issue of functoriality and a representation being cohomological.

5.1. **Cohomological representations of $\mathrm{GL}_n(\mathbb{R})$.** In this section we set up the context of cohomological representations. This is entirely standard material; we refer the reader to Borel-Wallach [6] and Schwermer [39] for generalities on the cohomology of representations.

We let $G_n = \mathrm{GL}_n$ and B_n be the standard Borel subgroup of upper triangular matrices in G_n. Let T_n be the diagonal torus in G_n and Z_n be the center of G_n. We denote by $X^+(T_n)$ the dominant (with respect to B_n) algebraic characters of T_n. For $\mu \in X^+(T_n)$ let (ρ_μ, M_μ) be the irreducible representation of $G_n(\mathbb{R})$ with highest weight μ. The Lie algebra of $G_n(\mathbb{R})$ will be denoted by $\mathfrak{g}_n$. We let $K_n = \mathrm{O}_n(\mathbb{R})Z_n(\mathbb{R})$ and K_n° be the topological connected component of the identity element in K_n.

Let $\mathrm{Coh}(G_n, \mu)$ be the set of all cuspidal automorphic representations $\pi = \otimes'_{p\leq\infty}\pi_p$ of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ such that

$$H^*(\mathfrak{g}_n, K_n^\circ\ ;\ \pi_\infty \otimes \rho_\mu) \neq (0).$$

By $H^*(\mathfrak{g}_n, K_n^\circ; -)$ we mean relative Lie algebra cohomology. We recall the following from [6, §I.5.1]: Given a $(\mathfrak{g}_n, K_n)$ module σ, one can talk about $H^*(\mathfrak{g}_n, K_n^\circ; \sigma)$ as well as $H^*(\mathfrak{g}_n, K_n; \sigma)$. Note that $K_n/K_n^\circ \simeq \mathbb{Z}/2\mathbb{Z}$ acts on $H^*(\mathfrak{g}_n, K_n^\circ; \sigma)$ and by taking invariants under this action we get $H^*(\mathfrak{g}_n, K_n; \sigma)$.

Observe that a global representation being cohomological is entirely a function of the representation at infinity. There are two very basic problems, one local and the other global, which has given rise to an enormous amount of literature on this theme.

1. (1) The local problem is to classify all irreducible admissible representations π_∞ of $G_n(\mathbb{R})$ which are cohomological, i.e., $H^*(\mathfrak{g}_n, K_n^\circ; \pi_\infty \otimes \rho_\mu) \neq (0)$, and for such representations to actually calculate the cohomology spaces.
2. (2) The global problem is to construct global cuspidal representations whose representation at infinity is cohomological in the above sense.

The reader is referred to Borel-Wallach [6] as a definitive reference for the local problem. For the purposes of this article we discuss the solution of the local problem for tempered representations of $\mathrm{GL}_n(\mathbb{R})$. To begin, we record a very simplified version of [6, Theorem II.5.3] and [6, Theorem II.5.4].

Theorem 5.1. *Let G be a reductive Lie group. Let K be a maximal compact subgroup adjoined with the center of G. Discrete series representations of G (if they exist) are cohomological and have nonvanishing cohomology only in degree* $\dim(G/K)/2$.

We have suppressed any mention of the finite dimensional coefficients because Wigner's Lemma [6, Theorem I.4.1] gives a necessary condition for the infinitesimal character, and nonvanishing cohomology of a representation pins down the finite dimensional representation. Here is a well known example illustrating this theorem.

Example 5.2. Let $G = G_2(\mathbb{R})$ and $K = K_2 = \mathrm{O}_2(\mathbb{R})Z_2(\mathbb{R})$. For any integer $l \geq 1$, we let M_l denote the irreducible representation of G of dimension l which is the $(l-1)$-th symmetric power of the standard two dimensional representation. Let D_l be the discrete series representation of lowest weight $l+1$. (If we take a weight k holomorphic cusp form then the representation at infinity is D_{k-1}.) The Langlands parameter of D_l is $\mathrm{Ind}_{\mathbb{C}^*}^{W_\mathbb{R}}(\chi_l)$, where $W_\mathbb{R}$ is the Weil group of $\mathbb{R}$, and χ_l is the character of $\mathbb{C}^*$ sending z to $(z/|z|)^l$. The representation D_l is cohomological; more precisely, we have

$$H^q(\mathfrak{g}, K; (D_l \otimes |\cdot|_\mathbb{R}^{-(l-1)/2}) \otimes M_l) = \begin{cases} \mathbb{C} & \text{if } q = 1, \\ 0 & \text{if } q \neq 1. \end{cases}$$

See [45, Proposition I.4 (1)] for instance. To compare our notation to the notation therein, take $h = l+1$, $a = l-1$, $\epsilon = 0$ and put $d = (h, a, \epsilon)$. Then our M_l is the $r[d]$ of [45] and our $D_l \otimes |\cdot|_\mathbb{R}^{-(l-1)/2}$ is the $\pi[d]$ of [45]. See [25, §2.1] for an SL_2 version of this example.

It is a standard fact that relative Lie algebra cohomology satisfies a Künneth rule [6, §I.1.3]. Using this one can see that if G is a product of m copies of $\mathrm{GL}_2(\mathbb{R})$ then the representation $D_{l_1} \otimes \cdots \otimes D_{l_m}$ is, up to twisting by a suitable power of $|\cdot|_\mathbb{R}$, cohomological with respect to the finite dimensional coefficients $M_{l_1} \otimes \cdots \otimes M_{l_m}$.

We now recall, very roughly, a version of Shapiro's lemma for relative Lie algebra cohomology. Consider a parabolically induced representation. The cohomology of the induced representation can be described in terms of the cohomology of the inducing representation. (See [6, Theorem III.3.3, (ii)] for a precise formulation.)

We can now give a reasonably complete picture for tempered representations of $\mathrm{GL}_n(\mathbb{R})$ which are cohomological. See Clozel [8, Lemme 3.14]. We follow the presentation in [27, §3.1].

Let $\mathcal{L}_0^+(G_n)$ stand for the set of all pairs $(w, \mathbf{l})$, with $\mathbf{l} = (l_1, \ldots, l_n) \in \mathbb{Z}^n$ such that $l_1 > \cdots > l_{[n/2]} > 0$ and $l_i = -l_{n-i+1}$, and $w \in \mathbb{Z}$, such that

$$w + \mathbf{l} \equiv \begin{cases} 1 & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

This set $\mathcal{L}_0^+(G_n)$ will parametrize certain tempered representations defined as follows. For $(w, \mathbf{l}) \in \mathcal{L}_0^+(G_n)$, define the parabolically induced representation $J(w, \mathbf{l})$ by

$$J(w, \mathbf{l}) = \mathrm{Ind}_{P_{2,\ldots,2}}^{G_n}((D_{l_1} \otimes |\cdot|_\mathbb{R}^{w/2}) \otimes \cdots \otimes (D_{l_{n/2}} \otimes |\cdot|_\mathbb{R}^{w/2}))$$

if n is even, and

$$J(w, \mathbf{l}) = \mathrm{Ind}_{P_{2,\ldots,2,1}}^{G_n}((D_{l_1} \otimes |\cdot|_\mathbb{R}^{w/2}) \otimes \cdots \otimes (D_{l_{(n-1)/2}} \otimes |\cdot|_\mathbb{R}^{w/2}) \otimes |\cdot|_\mathbb{R}^{w/2})$$

if n is odd. It is well known that, up to the twist $|\cdot|_{\mathbb{R}}^{w/2}$, the representations $J(w,\mathbf{l})$ are irreducible tempered representations of G_n [22, §2].

Now we describe the finite dimensional coefficients. Let $X_0^+(T_n)$ stand for all dominant integral weights $\mu = (\mu_1,\ldots,\mu_n)$ satisfying the purity condition that there is an integer w, called the weight of μ, such that $\mu_i + \mu_{n-i+1} = w$. The sets $\mathcal{L}_0^+(G_n)$ and $X_0^+(T_n)$ are in bijection via the map $(w,\mathbf{l}) \mapsto \mu = w/2 + \mathbf{l}/2 - \rho_n$ where ρ_n is half the sum of positive roots for GL_n. Let w_n be the Weyl group element of G_n of longest length and let $\mu^\vee = -w_n \cdot \mu$. Then $\rho_{\mu^\vee} \simeq (\rho_\mu)^\vee$ is the contragredient of ρ_μ.

Assume that the pair $(w,\mathbf{l})$ corresponds to μ as above. Using Example 5.2 on the cohomology of discrete series representations, and appealing to the Künneth rule and Shapiro's lemma as recalled above, one can conclude that

$$H^q(\mathfrak{g}_n, K_n^\circ; (J(w,\mathbf{l}) \otimes \mathrm{sgn}^t) \otimes M_{\mu^\vee}) = (0)$$

unless the degree q is in the so-called cuspidal range $b_n \leq q \leq t_n$, where the bottom degree b_n is given by

$$b_n = \begin{cases} n^2/4 & \text{if } n \text{ is even,} \\ (n^2-1)/4 & \text{if } n \text{ is odd,} \end{cases}$$

and the top degree t_n is given by

$$t_n = \begin{cases} ((n+1)^2-1)/4 - 1 & \text{if } n \text{ is even,} \\ (n+1)^2/4 - 1 & \text{if } n \text{ is odd,} \end{cases}$$

and finally that the dimension of $H^q(\mathfrak{g}_n, K_n; (J(w,\mathbf{l}) \otimes \mathrm{sgn}^t) \otimes M_{\mu^\vee})$ is 1 if $q = b_n$ or $q = t_n$. The exponent t of the sign character sgn is in $\{0,1\}$. If n is even, t plays no role since $J(w,\mathbf{l}) \otimes \mathrm{sgn} = J(w,\mathbf{l})$. If n is odd, t is determined by the weight of μ and the parity of $(n-1)/2$, due to considerations of central character (Wigner's lemma).

To complete the picture one notes that, given M_μ, there is, up to twisting by the sign character, only one irreducible, unitary (up to twisting by $|\cdot|_{\mathbb{R}}^{-w/2}$), generic representation with nonvanishing cohomology with respect to M_μ and this representation is a suitable $J(w,\mathbf{l})$. (See [27, §3.1.3].)

Remark 5.3. Let π be a cohomological cuspidal algebraic ([8, §1.2.3]) automorphic representation of $G_n(\mathbb{A}_\mathbb{Q})$ then the representation π_∞ at infinity has to be a $J(w,\mathbf{l})$ for some $(w,\mathbf{l}) \in \mathcal{L}_0^+(G_n)$. This can be seen as follows. Since π is cuspidal and algebraic, by the purity lemma [8, Lemme 4.9], we get that the parameter of π_∞ is pure. Since it is cohomological the finite dimensional coefficients has a highest weight μ which is also pure, i.e., $\mu \in X_0^+(T_n)$. Further, π_∞ being generic and essentially unitary implies that it is a $J(w,\mathbf{l})$ as above.

Example 5.4. To illuminate this picture we work through the above recipe for the case of a holomorphic cusp form. (We use the notation introduced in Example 5.2 and the previous sections.) Let $\varphi \in S_k(N,\omega)$ and consider the cuspidal automorphic representation $\pi = \pi(\varphi) \otimes |\cdot|^s$. Then $\pi_\infty = \pi(\varphi)_\infty \otimes |\cdot|_{\mathbb{R}}^s = D_{k-1} \otimes |\cdot|_{\mathbb{R}}^s$.

(1) If k is even, then the representation π_∞ is a $J(w, \mathbf{l})$ exactly when $w = 2s \in \mathbb{Z}$ and $w + k - 1$ is odd. Hence $s \in \mathbb{Z}$ and $\pi_\infty = J(2s, (k-1, -(k-1)))$. The corresponding dominant weight μ is $(s + (k-2)/2, s - (k-2)/2)$. The representation $M_{\mu^\vee}$ is $M_{k-1} \otimes (\det)^{-s-(k-2)/2}$. (For a dominant weight $\mu = (\mu_1, \mu_2) \in \mathbb{Z}^2$ the rational representation $M_{\mu^\vee}$ is $M_{\mu_1-\mu_2+1} \otimes (\det)^{-\mu_1}$.) Using the fact that $\det = \mathrm{sgn} \otimes |\cdot|_{\mathbb{R}}$ and that $D_{k-1} \otimes \mathrm{sgn} = D_{k-1}$, we get

$$\begin{aligned} \pi_\infty \otimes M_{\mu^\vee} &= (D_{k-1} \otimes |\cdot|_{\mathbb{R}}^{s}) \otimes (M_{k-1} \otimes (\det)^{-s-(k-2)/2}) \\ &= (D_{k-1} \otimes |\cdot|_{\mathbb{R}}^{-(k-2)/2}) \otimes M_{k-1}, \end{aligned}$$

which has nontrivial $(\mathfrak{g}, K)$-cohomology (see Example 5.2).

(2) If $k \geq 3$ is odd, then π_∞ is a $J(w, \mathbf{l})$ exactly when $w = 2s$ is an odd integer. Letting $s = 1/2 + r$, with $r \in \mathbb{Z}$, we have $\pi_\infty = J(2r+1, (k-1, -(k-1)))$. The corresponding dominant weight μ is $(r + (k-1)/2, r + 1 - (k-1)/2)$. The representation $M_{\mu^\vee}$ is $M_{k-1} \otimes (\det)^{-r-(k-1)/2}$. In this case we get

$$\begin{aligned} \pi_\infty \otimes M_{\mu^\vee} &= (D_{k-1} \otimes |\cdot|_{\mathbb{R}}^{1/2+r}) \otimes (M_{k-1} \otimes (\det)^{-r-(k-1)/2}) \\ &= (D_{k-1} \otimes |\cdot|_{\mathbb{R}}^{-(k-2)/2}) \otimes M_{k-1}, \end{aligned}$$

which has nontrivial $(\mathfrak{g}, K)$-cohomology as mentioned before. We have excluded the case $k = 1$, because, firstly, the representation at infinity is not cohomological, and secondly, any symmetric power L-function of a weight one form has no critical points.

We finally remark that in both cases, the condition π_∞ being a $J(w, \mathbf{l})$ is exactly the condition which ensures that the representation $\pi = \pi(\varphi) \otimes |\cdot|^s$ is regular algebraic in the sense of Clozel [8, §1.2.3 and §3.4].

5.2. **Functoriality and cohomological representations.** Now we turn to the global problem, namely, to construct a cuspidal automorphic representation whose representation at infinity is cohomological. The specific theorem we are interested in is the following.

Theorem 5.5. *Let $\varphi \in S_k(N, \omega)$ with $k \geq 2$. Let $n \geq 1$. Assume that $\mathrm{Sym}^n(\pi(\varphi))$ is a cuspidal representation of $\mathrm{GL}_{n+1}(\mathbb{A}_{\mathbb{Q}})$. Let*

$$\Pi = \mathrm{Sym}^n(\pi(\varphi)) \otimes \xi \otimes |\cdot|^s$$

where ξ is any idèle class character such that $\xi_\infty = \mathrm{sgn}^\epsilon$, with $\epsilon \in \{0, 1\}$, and $|\cdot|$ is the adèlic norm. We suppose that s and ϵ satisfy:

(1) *If n is even, then let $s \in \mathbb{Z}$ and $\epsilon \equiv n(k-1)/2 \pmod 2$.*
(2) *If n is odd then, we let $s \in \mathbb{Z}$ if k is even, and we let $s \in 1/2 + \mathbb{Z}$ if k is odd. We impose no condition on ϵ.*

Then $\Pi \in \mathrm{Coh}(G_{n+1}, \mu^\vee)$ *where* $\mu \in X_0^+(T_{n+1})$ *is given by*

$$\mu = \left(\frac{n(k-2)}{2} + s, \frac{(n-2)(k-2)}{2} + s, \dots, \frac{-n(k-2)}{2} + s\right) = (k-2)\rho_{n+1} + s.$$

(Recall that ρ_{n+1} *is half the sum of positive roots of* GL_{n+1}*.) In other words, the representation* $\mathrm{Sym}^n(\pi(\varphi))\otimes\xi\otimes|\cdot|^s$*, with* ξ *and* s *as above, contributes to the cohomology of the locally symmetric space* $\mathrm{GL}_{n+1}(\mathbb{Q})\backslash\mathrm{GL}_{n+1}(\mathbb{A}_\mathbb{Q})/K_f K^\circ_{n+1,\infty}$ *with coefficients in the local system determined by* $\rho_{\mu^\vee}$*, where* K_f *is a deep enough open compact subgroup of* $\mathrm{GL}_{n+1}(\mathbb{A}_{\mathbb{Q},f})$*. (Here* $\mathbb{A}_{\mathbb{Q},f}$ *denotes the finite adèles of* $\mathbb{Q}$*.)*

Proof. See Labesse–Schwermer [25, Proposition 5.4] for an SL_n-version of this theorem. When $k = 2$, the theorem has also been observed by Kazhdan, Mazur and Schmidt [16, pp.99].

We sketch the details in the case when $n = 2r$ is even (the case when n is odd being absolutely similar.) The proof follows by observing that the representation at infinity of $\mathrm{Sym}^n(\pi(\varphi))$ is the representation of $\mathrm{GL}_{n+1}(\mathbb{R})$ whose Langlands parameter is $\mathrm{Sym}^n(\mathrm{Ind}_{\mathbb{C}^*}^{W_\mathbb{R}}(\chi_{k-1}))$ where $\chi_{k-1}(z) = (z/|z|)^{k-1}$. It is a pleasant exercise to calculate a symmetric power of a two dimensional induced representation, after doing which one gets that the representation Π_∞ is given by

$$\begin{aligned}\Pi_\infty &= \mathrm{Ind}_{P_{2,\dots,2,1}}^{G_{n+1}}(D_{2r(k-1)} \otimes \cdots \otimes D_{2(k-1)} \otimes \mathrm{sgn}^{r(k-1)}) \otimes \xi_\infty \otimes |\cdot|^s_\mathbb{R} \\ &= \mathrm{Ind}_{P_{2,\dots,2,1}}^{G_{n+1}}(D_{2r(k-1)} \otimes \cdots \otimes D_{2(k-1)} \otimes \mathrm{sgn}^{r(k-1)+\epsilon}) \otimes |\cdot|^s_\mathbb{R}.\end{aligned}$$

We deduce that Π_∞ is a $J(w, \mathbf{l})$ (which, as mentioned before, is equivalent to Π being regular algebraic) exactly when $w = 2s \in \mathbb{Z}$, $r(k-1) + \epsilon$ is even,

$$\mathbf{l} = (2r(k-1), \dots, 2(k-1), 0, -2(k-1), \dots, -2r(k-1)) = 2(k-1)\rho_{n+1},$$

and $w + \mathbf{l}$ is even which implies that w is even. These conditions are satisfied by the hypothesis in the theorem. The weight μ is determined by $\mu = w/2 + \mathbf{l}/2 - \rho_{n+1}$ and the first part of the theorem follows from the discussion in the previous section.

Finally, the relation with the cohomology of locally symmetric spaces follows as in [25, §1] or [27, §3.2]. □

One might view this theorem as an example of the possible dictum that a functorial lift of a cohomological representation is cohomological. (However, see Example 5.6 below.) This dictum has been used in many instances to construct global representations which contribute to cuspidal cohomology. The following is a sampling of such results–which by no means is to be considered exhaustive–to add weight to the above dictum.

(1) Labesse and Schwermer [25] proved the existence of nontrivial cuspidal cohomology classes for SL_2 and SL_3 over any number field E which contains a totally real number field F such that $F = F_0 \subset F_1 \subset \cdots \subset F_n = E$ with each each F_{i+1}/F_i either a cyclic extension of prime degree or a non-normal

cubic extension. The functorial lifts used were base change for GL_2 and the symmetric square lifting of Gelbart and Jacquet. This was generalized for SL_n over E, in conjunction with Borel [5]; with the additional input of base change for GL_n.

(2) Motivated by [25], Clozel [7] used automorphic induction and proved the existence of nontrivial cuspidal cohomology classes for SL_{2n} over any number field.

(3) Rajan [34], also motivated by [25], proved the existence of nontrivial cuspidal cohomology classes for $\mathrm{SL}_1(D)$ for a quaternion division algebra D over a number field E, with E being an extension of a totally real number field F with solvable Galois closure. Other than base change, he used the Jacquet–Langlands correspondence.

(4) Ash and Ginzburg [1, §4] have commented on a couple of examples of cuspidal cohomology classes for GL_4 over $\mathbb{Q}$. The first is by lifting from GSp_4 to GL_4 a weight 3 Seigel modular form. The second is to use automorphic induction from GL_2 over a quadratic extension.

(5) Ramakrishnan and Wang [37] used the lifting from $\mathrm{GL}_2 \times \mathrm{GL}_3 \to \mathrm{GL}_6$, due to Kim and Shahidi, to construct cuspidal cohomology classes of GL_6 over $\mathbb{Q}$.

In almost all the above works, functoriality is used to construct cuspidal representations, and in doing so, one exercises some control over the representations at infinity to arrange for them to be cohomological. It is an interesting question to ask if the converse of the above dictum is true, namely, if a lift is cohomological, then whether the preimage is, *a fortiori*, cohomological? (See Example 5.7 below.) We would like to draw attention to a conjecture of Clozel [7, §1] which is motivated by the ideas of Labesse and Schwermer. The conjecture roughly states that given a tempered cohomological representation at infinity, one can find a global cuspidal automorphic representation whose representation at infinity is the given one.

Example 5.6. We construct an example to show that a functorial lift of a cohomological representation need not be cohomological. For an even integer k, take two weight k holomorphic cusp forms φ_1 and φ_2, and let $\pi_i = \pi(\varphi_i)$ for $i = 1, 2$. By Example 5.4 we have that both π_1 and π_2 are cohomological representations. Put $\Pi = \pi_1 \boxtimes \pi_2$ (see Ramakrishnan [35]). Choose the forms φ_1 and φ_2 such that Π is cuspidal; this can be arranged by taking exactly one of them to be dihedral, or by arranging that π_1 is not $\pi_2 \otimes \chi$ for any character χ, by virtue of [36, Theorem 11.1]. It is easy to see that Π_∞ is given by

$$\Pi_\infty = \mathrm{Ind}_{P_{2,1,1}}^{G_4(\mathbb{R})}(D_{2(k-1)} \otimes \mathrm{sgn} \otimes \mathbb{1}),$$

where $\mathbb{1}$ is the trivial representation of $\mathbb{R}^*$. Observe that Π_∞ is not a $J(w, \mathbf{l})$ and hence is not cohomological by applying Remark 5.3. (Note that Π, as it stands, is not algebraic, but we can replace Π by $\pi_1 \overset{T}{\boxtimes} \pi_2$ (see [8, Definition 1.10]) and make it algebraic; this replaces Π_∞ by $\Pi_\infty \otimes |\cdot|_{\mathbb{R}}^{1/2}$.) However, note that if we took φ_1 and φ_2 to be in general position (unequal weights) then the lift Π would be cohomological. One should therefore think of the dictum that *functoriality preserves the property of being cohomological* only as a guiding principle rather than a precise conjecture. Similarly, it is possible to construct such an example for the lifting from $\mathrm{GL}_2 \times \mathrm{GL}_3$ to GL_6.

Example 5.7. We would like to mention that in the converse direction the $\mathrm{GL}_2 \times \mathrm{GL}_2$ to GL_4 lifting is well behaved. Now let φ_i have weight $k_i \geq 1$, for $i = 1, 2$, and assume without loss of generality that $k_1 \geq k_2$. With $\Pi = \pi(\varphi_1) \boxtimes \pi(\varphi_2)$ we have

$$\Pi_\infty = \mathrm{Ind}_{P_{2,2}}^{G_4(\mathbb{R})}(D_{k_1+k_2-2} \otimes D_{k_1-k_2}).$$

Suppose Π_∞ is cohomological, i.e., is a $J(w, \mathbf{l})$, then we would have $k_1 + k_2 - 2 > k_1 - k_2 > 0$, which implies that $k_1 > k_2 \geq 2$, and hence both $\pi(\varphi_1)$ and $\pi(\varphi_2)$ are cohomological.

6. Special values of L-functions of GL_n: The work of Mahnkopf

6.1. **General remarks on functoriality and special values.** This section is a summary of some recent results due to Joachim Mahnkopf [26] [27]. In this work he proves certain special values theorems for the standard L-functions of cohomological cuspidal automorphic representations of GL_n. In principle one can appeal to functoriality and this work of Mahnkopf to prove new special values theorems. For example, given a cusp form $\varphi \in S_k(N, \omega)$, let $\pi(\varphi)$ denote the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Functoriality predicts the existence of an automorphic representation $\mathrm{Sym}^n(\pi(\varphi))$ of $\mathrm{GL}_{n+1}(\mathbb{A}_\mathbb{Q})$. (See §2.) Then it is easy to check that

$$L(s, \mathrm{Sym}^n(\pi(\varphi)) = L(s + n(k-1)/2, \mathrm{Sym}^n\varphi),$$

where the left hand side is the standard L-function of $\mathrm{Sym}^n(\pi(\varphi))$. Using Mahnkopf's work for the function on the left, one can hope to prove a special values theorem for the function on the right. This is fine in principle, but there are several obstacles to overcome before it can be made to work.

6.2. **The main results of Mahnkopf** [27]**.** Let $\mu \in X_0^+(T_n)$ and let $\pi \in \mathrm{Coh}(G_n, \mu)$. We let $L(s, \pi) = \prod_{p \leq \infty} L(s, \pi_p)$ be the standard L-function attached to π. Any character χ_∞ of $\mathbb{R}^*$ is of the form $\chi_\infty = \epsilon_\infty |\cdot|^m$ for a complex number m. We say χ_∞ is critical for π_∞ if

(1) $m \in 1/2 + \mathbb{Z}$ if n is even, and $m \in \mathbb{Z}$ if n is odd; and
(2) $L(\pi_\infty \otimes \chi_\infty, 0)$ and $L(\pi_\infty^\vee \otimes \chi_\infty^{-1}, 1)$ are regular values.

We say $\chi : \mathbb{Q}^* \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^*$ is critical for π if χ_∞ is critical for π_∞. Let $\mathrm{Crit}(\pi)$ stand for all such characters χ which are critical for π. Let $\mathrm{Crit}(\pi)^\leq$ stand for all $\chi \in \mathrm{Crit}(\pi)$ such that if $\chi_\infty = \epsilon_\infty |\cdot|^m$ then $m \leq (1 - \mathrm{wt}(\mu))/2$.

Let $\pi \in \mathrm{Coh}(G_n,\mu)$ and let $\chi \in \mathrm{Crit}(\pi)$. Let $\chi_\infty = \epsilon_\infty|\cdot|^m$. Given $\mu = (\mu_1,\dots,\mu_n) \in X^+(T_n)$ choose a $\lambda = (\lambda_1,\dots,\lambda_{n-1}) \in X^+(T_{n-1})$ such that

(1) $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_n$; and
(2) $\lambda_{n/2} = -m+1/2$ if n is even, and $\lambda_{(n+1)/2} = -m$ if n is odd.

Proposition 1.1 of [26] says that such a λ exists. Let P be the standard parabolic subgroup of G_{n-1} of type $(n-2,1)$ and let W^P be a system of representatives for $W_{M_P}\backslash W_{G_{n-1}}$. Let $\widehat{w} \in W^P$ be given by

$$\widehat{w} = \begin{pmatrix} 1 & 2 & \cdots & \left[\frac{n}{2}\right]-1 & \left[\frac{n}{2}\right] & \left[\frac{n}{2}\right]+1 & \cdots & n-1 \\ 1 & 2 & \cdots & \left[\frac{n}{2}\right]-1 & n-1 & \left[\frac{n}{2}\right] & \cdots & n-2 \end{pmatrix}.$$

Define the weight $\mu' = (\widehat{w}(\lambda+\rho_{n-1})-\rho_{n-1})|_{T_{n-2}} \in X^+(T_{n-2})$ where T_{n-2} is embedded in T_{n-1} as $t \mapsto \mathrm{diag}(t,1)$.

Theorem 6.1 (Theorem 5.4 in Mahnkopf [27])**.** *Let $\mu \in X_0^+(T_n)$ be regular and let $\pi \in \mathrm{Coh}(\mathrm{GL}_n,\mu^\vee)$. Let $\mu' \in X^+(T_{n-2})$ be as above and $\pi' \in \mathrm{Coh}(\mathrm{GL}_{n-2},\mu')$; if n is odd then π' has to satisfy a parity condition. We have*

(1) $\mathrm{Crit}(\pi)^{\leq} \subset \mathrm{Crit}(\pi')^{\leq}$.
(2) *Let $\chi \in \mathrm{Crit}(\pi)^{\leq}$, with $\chi_\infty = \epsilon_\infty|\cdot|^m$. There exists a collection of complex numbers $\Omega(\pi,\pi',\epsilon_\infty) \in \mathbb{C}^*/\mathbb{Q}(\pi)\mathbb{Q}(\pi')$ such that for any finite extension $E/\mathbb{Q}(\pi)\mathbb{Q}(\pi')$ the tuple $\{\Omega(\pi,\pi',\epsilon_\infty)\}_{\sigma\in\mathrm{Hom}(E,\mathbb{C})} \in (E\otimes\mathbb{C})^*/(\mathbb{Q}(\pi)\mathbb{Q}(\pi'))^*$ is well defined. There exists a complex number $P_\mu(m)$, depending only on μ and m, subject to Assumption 6.2 below, such that for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ and almost all χ as above, we have*

$$\left(\frac{\mathfrak{g}(\chi)\mathfrak{G}(\eta)P_\mu(m)}{\Omega(\pi,\pi',\epsilon_\infty)}\frac{L(\pi\otimes\chi\eta,0)}{L(\pi'^\vee\otimes\chi,0)}\right)^\sigma = \frac{\mathfrak{g}(\chi^\sigma)\mathfrak{G}(\eta^\sigma)P_\mu(m)}{\Omega(\pi^\sigma,\pi'^\sigma,\epsilon_\infty)}\frac{L(\pi^\sigma\otimes\chi^\sigma\eta^\sigma,0)}{L((\pi'^\vee)^\sigma\otimes\chi^\sigma,0)},$$

where η is a certain auxiliary character and $\mathfrak{G}(\eta)$ a certain product of Gauss sums associated to η.

The above theorem is valid only under the following assumption.

Assumption 6.2. $P_\mu(m) \neq 0$.

The quantity $P_\mu(m)$ is the value at $s = 1/2$ of an archimedean Rankin–Selberg integral attached to certain cohomological choice of Whittaker functions. Mahnkopf proves a necessary condition for this nonvanishing assumption [27, §6]. At present this seems to be a serious limitation of this technique. It is widely believed that this assumption is valid and it has shown up in several other works based on the same, or at any rate similar, techniques. See for instance Ash–Ginzburg [1], Kazhdan-Mazur-Schmidt [16] and Harris [13]. *It is an important technical problem to be able to prove this nonvanishing hypothesis.*

The proof of the above theorem combines both the Langlands–Shahidi and the Rankin–Selberg methods of studying L-functions. One considers the pair of representations $\pi \times \mathrm{Ind}_P^{G_{n-1}}(\pi'\otimes\chi)$ of $G_n(\mathbb{A}_\mathbb{Q}) \times G_{n-1}(\mathbb{A}_\mathbb{Q})$ and carefully chooses a cusp form $\phi \in \pi$ and an Eisenstein series $\mathcal{E}$ corresponding to a section in $\mathrm{Ind}_P^{G_{n-1}}(\pi' \otimes \chi)$. To this pair $(\phi, \mathcal{E})$ a certain Rankin–Selberg type zeta integral [27, 2.1.2], which has a cohomological interpretation, computes the quotient of L-functions appearing in the theorem.

The theorem roughly says that the special values of a standard L-function for GL_n are determined in terms of those of a standard L-function for GL_{n-2}. This descent process terminates since we know the special values of L-functions for GL_1 and GL_2, and we get the following theorem; see [27, §5.5] for making the right choices in the induction on n.

Theorem 6.3 (Theorem A in Mahnkopf [27])**.** *Assume that $\mu \in X_0^+(T_n)$ is regular and let $\pi \in \mathrm{Coh}(G_n, \mu^\vee)$. Let $\chi \in \mathrm{Crit}(\pi)^{\leq}$. To π and χ_∞ is attached $\Omega(\pi, \chi_\infty) \in \mathbb{C}$ such that for all but finitely many such χ we have*

$$\left(\frac{\mathfrak{g}(\chi)^{[n/2]}\mathfrak{G}(\eta)}{\Omega(\pi, \chi_\infty)} L(\pi \otimes \chi\eta, 0) \right)^\sigma = \frac{\mathfrak{g}(\chi^\sigma)^{[n/2]}\mathfrak{G}(\eta^\sigma)}{\Omega(\pi^\sigma, \chi_\infty^\sigma)} L(\pi^\sigma \otimes \chi^\sigma\eta^\sigma, 0),$$

where η is a certain auxiliary character and $\mathfrak{G}(\eta)$ a certain product of Gauss sums associated to η. Moreover, write $\chi_\infty = \epsilon'_\infty |\cdot|_\infty^l$ and set $\epsilon(\chi_\infty) = \epsilon'_\infty \mathrm{sgn}^l$. There are periods $\Omega_\epsilon(\pi) \in \mathbb{C}^*$ if n is even, and $\Omega(\pi) \in \mathbb{C}^*$ if n is odd, and a collection $P_\mu^l \in \mathbb{C}$, such that $\Omega(\pi, \chi_\infty) = P_\mu^l \Omega(\pi)$ if n is odd, and $\Omega(\pi, \chi_\infty) = P_\mu^l \Omega_{\epsilon(\chi_\infty)}(\pi)$ if n is even.

Note that Theorem 6.3, since it uses Theorem 6.1, also depends on Assumption 6.2.

7. A conjecture on twisted L-functions

The periods c^+ and c^- which appear in Deligne's conjecture are motivically defined. (See Deligne [9, (1.7.2)].) On the other hand, the periods which appear in the work of Harder, and also Mahnkopf, have an entirely different origin, namely, they come by a comparison of rational structures on cuspidal cohomology on the one hand and a Whittaker model for the representation, on the other. See Harder [12, p. 81] and Mahnkopf [27, §3.4]. It is not at the moment clear how one might explicitly compare these different periods attached to the same object. (See also Remark (2) in Harder's paper [12, p. 85].)

However, one might ask if these different periods behave in the same manner under twisting. Here is a simple example to illustrate this. Let χ be an even Dirichlet character. Let m be an even positive integer. Such an m is critical for $L(s, \chi)$. It is well known [30, Corollary VII.2.10] that

$$L_f(m, \chi) \sim_{\mathbb{Q}(\chi)} (2\pi i)^m \mathfrak{g}(\chi).$$

By $\sim_{\mathbb{Q}(\chi)}$ we mean up to an element of the (rationality) field $\mathbb{Q}(\chi)$ generated by the values of χ. Now let η be possibly another even Dirichlet character. Applying the result to the character $\chi\eta$, and using [41, Lemma 8], we get

$$L_f(m, \chi\eta)/L_f(m, \chi) \sim_{\mathbb{Q}(\chi)\mathbb{Q}(\eta)} \mathfrak{g}(\eta).$$

Observe that the *period*, namely the $(2\pi i)^m$, does not show up, and we have the relation that the special value of the twisted L-function and the original L-function differ, up to rational quantities, by the Gauss sum of the twisting character.

Another example along these lines which follows from Shimura [42] is the following. Let $\varphi \in S_k(N,\omega)$ and let η be an even Dirichlet character. For any integer m, with $1 \leq m \leq k-1$, we have

$$L_f(m,\varphi,\eta) \sim_{\mathbb{Q}(\varphi)\mathbb{Q}(\eta)} \mathfrak{g}(\eta) L_f(m,\varphi).$$

The point being that, in the above relation, the periods $c^{\pm}(\varphi)$ do not show up, and so the definition of these periods is immaterial. (One can rewrite this relation entirely in terms of periods of the associated motives and it takes the form $c^{\pm}(M(\varphi)\otimes M(\eta)) \sim \mathfrak{g}(\eta)c^{\pm}(M(\varphi))$, the notation being obvious.)

Even if one cannot prove a precise theorem on special values of L-functions in terms of these–motivically or otherwise defined–periods, one can still hope to prove such period relations. Sometimes such period relations are sufficient for applications; see for instance Murty–Ramakrishnan [29] where such a period relation is used to prove Tate's conjecture in a certain case.

With this motivation, we formulate the following conjecture on the behavior of the special values of symmetric power L-functions under twisting by Dirichlet characters.

Conjecture 7.1. *Let $\varphi \in S_k(N,\omega)$ be a primitive form. Let η be a primitive Dirichlet character.*

(1) *Suppose η is even, i.e., $\eta(-1)=1$. Then the critical set for $L_f(s,\mathrm{Sym}^n\varphi,\eta)$ is the same as the critical set for $L_f(s,\mathrm{Sym}^n\varphi)$, and if m is critical, then*

$$L_f(m,\mathrm{Sym}^n\varphi,\eta) \sim \mathfrak{g}(\eta)^{\lceil (n+1)/2\rceil} L_f(m,\mathrm{Sym}^n\varphi),$$

unless n is even and m is odd (to the left of center of symmetry), in which case we have

$$L_f(m,\mathrm{Sym}^n\varphi,\eta) \sim \mathfrak{g}(\eta)^{n/2} L_f(m,\mathrm{Sym}^n\varphi).$$

(2) *Suppose η is odd, i.e., $\eta(-1)=-1$, and n is even. Then, if m is critical for $L_f(s,\mathrm{Sym}^n\varphi,\eta)$, then either $m+1$ or $m-1$ is critical for $L_f(s,\mathrm{Sym}^n\varphi)$. For such an m to the right of the center of symmetry we have*

$$L_f(m,\mathrm{Sym}^n\varphi,\eta) \sim ((2\pi i)^{\mp}\mathfrak{g}(\eta))^{n/2+1} L_f(m\pm 1,\mathrm{Sym}^n\varphi),$$

and if m is to the left of the center of symmetry, we have

$$L_f(m,\mathrm{Sym}^n\varphi,\eta) \sim ((2\pi i)^{\mp}\mathfrak{g}(\eta))^{n/2} L_f(m\pm 1,\mathrm{Sym}^n\varphi).$$

(3) *Suppose η is odd, i.e., $\eta(-1) = -1$, and n is odd. Then the critical set for $L_f(s, \mathrm{Sym}^n\varphi, \eta)$ is the same as the critical set for $L_f(s, \mathrm{Sym}^n\varphi)$. Let $k \geq 3$. If m is critical for $L_f(s, \mathrm{Sym}^n\varphi, \eta)$, then either $m + 1$ or $m - 1$ is critical for $L_f(s, \mathrm{Sym}^n\varphi)$, and for such an m*

$$L_f(m, \mathrm{Sym}^n\varphi, \eta) \sim ((2\pi i)^{\mp}\mathfrak{g}(\eta))^{(n+1)/2} L_f(m \pm 1, \mathrm{Sym}^n\varphi).$$

In all the three cases $\sim$ means up to an element of $\mathbb{Q}(\varphi)\mathbb{Q}(\eta)$.

Now we elaborate on the heuristics on which we formulated the above conjecture. For $n = 1$ and $n = 2$ this is contained in the theorems of Shimura [41] [42] and Sturm [43] [44] respectively. For $n = 3$, using results on triple product L-functions for which Blasius [2] is a convenient reference and using Garrett–Harris [10, §6], one can verify that the above conjecture is true. Further, for $n \geq 4$ and if φ is dihedral, i.e., $\pi(\varphi) = \mathrm{AI}_{K/\mathbb{Q}}(\chi)$, then the conjecture follows by applying the known cases of $n = 1, 2$ to each summand in the isobaric decomposition in Lemma 4.1. Observe that the exponent $\lceil (n + 1)/2 \rceil$ appearing in the conjecture is the number of summands in the isobaric decomposition.

We leave it to the reader to check that the above conjecture is compatible with conjectures of Blasius [3, Conjecture L.9.8] and Panchiskin [32, Conjecture 2.3] on the behavior of periods of motives twisted by Artin motives.

It appears that the authors can prove this conjecture, at least in part, and so really prove a relation amongst appropriate periods, using Theorem 6.3 of Mahnkopf; at least in the case when $\mathrm{Sym}^n(\pi(\varphi))$ is known to exist as a cuspidal automorphic representation.

References

[1] A. Ash and D. Ginzburg, *p-adic L-functions for* $\mathrm{GL}(2n)$. Invent. Math. 116, no. 1-3, 27–73 (1994).

[2] D. Blasius, *Critical values of certain tensor product L-functions,* Appendix to Orloff [31].

[3] D. Blasius, *Period relations and critical values of L-functions.* Olga Taussky-Todd: in memoriam. Pacific J. Math., Special Issue, 53–83 (1997).

[4] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations.* Proc. Sympos. Pure Math., XXXIII, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, pp. 189–207, Amer. Math. Soc., Providence, R.I., (1979).

[5] A. Borel, J.-P. Labesse and J. Schwermer, *On the cuspidal cohomology of S-arithmetic subgroups of reductive groups over number fields.* Compositio Math. 102, no. 1, 1–40 (1996).

[6] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups.* Second edition. Mathematical Surveys and Monographs, 67. American Mathematical Society, Providence, RI, 2000. xviii+260 pp.

[7] L. Clozel, *On the cuspidal cohomology of arithmetic subgroups of* $\mathrm{SL}(2n)$ *and the first Betti number of arithmetic* 3*-manifolds.* Duke Math. J. 55, no. 2, 475–486 (1987).

[8] L. Clozel, Motifs et formes automorphes: applications du principe de fonctorialité.(French) [Motives and automorphic forms: applications of the functoriality principle] Automorphic

forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), 77–159, Perspect. Math., 10, Academic Press, Boston, MA, 1990.

[9] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales (French),* With an appendix by N. Koblitz and A. Ogus. Proc. Sympos. Pure Math., XXXIII, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 313–346, Amer. Math. Soc., Providence, R.I., 1979.

[10] P. Garrett and M. Harris, *Special values of triple product L-functions.* American J. Math., Vol 115, 159–238 (1993).

[11] S. Gelbart and H. Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3). Ann. Sci. École Norm. Sup. (4) 11, no. 4, 471–542 (1978).

[12] G. Harder, *General aspects in the theory of modular symbols.* Seminar on number theory, Paris 1981–82 (Paris, 1981/1982), 73–88, Progr. Math., 38, Birkhuser Boston, Boston, MA, 1983.

[13] M. Harris, *Occult period invariants and critical values of the degree four L-function of* GSp(4). Contributions to automorphic forms, geometry, and number theory, 331–354, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[14] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties.* With an appendix by Vladimir G. Berkovich. Annals of Mathematics Studies, 151. Princeton University Press, Princeton, NJ, 2001.

[15] G. Henniart, *Une preuve simple des conjectures de Langlands pour* GL(n) *sur un corps p-adique. (French) [A simple proof of the Langlands conjectures for* GL(n) *over a p-adic field]* Invent. Math. 139, no. 2, 439–455 (2000).

[16] D. Kazhdan, B. Mazur and C.-G. Schmidt *Relative modular symbols and Rankin–Selberg convolutions.* J.reine angew. Math. 519, 97–141 (2000).

[17] H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. J. Amer. Math. Soc. 16 (2003), no. 1, 139–183 (electronic).

[18] H. Kim *An example of non-normal quintic automorphic induction and modularity of symmetric powers of cusp forms of icosahedral type.* Invent. Math. 156, no. 3, 495–502 (2004).

[19] H. Kim and F. Shahidi, *Holomorphy of Rankin triple L-functions; special values and root numbers for symmetric cube L-functions.* Israel Jour. of Math., 120, 449–466 (2000).

[20] H. Kim and F. Shahidi, *Symmetric cube L-functions for* GL_2 *are entire.* Ann. of Math., (2) 150, no. 2, 645–662 (1999).

[21] H. Kim and F. Shahidi, *Functorial products for* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *and the symmetric cube for* GL_2. *With an appendix by Colin J. Bushnell and Guy Henniart.* Ann. of Math. (2) 155, no. 3, 837–893 (2002).

[22] A. Knapp, *Local Langlands correspondence: the archimedean case,* Motives (Seattle, WA, 1991), 393–410, Proc. Sympos. Pure Math., 55, Part II, Amer. Math. Soc., Providence, RI, 1994.

[23] S. Kudla, *The local Langlands correspondence : The non-archimedean case,* Proc. Symp. Pure Math., vol 55, part II, 365-391 (1994).

[24] P.C. Kutzko, *The Langlands conjecture for* Gl_2 *of a local field.* Ann. of Math. (2) 112, no. 2, 381–412 (1980).

[25] J.-P. Labesse and J. Schwermer, *On liftings and cusp cohomology of arithmetic groups.* Invent. Math., 83, 383–401 (1986).

[26] J. Mahnkopf, *On the special values of automorphic L-functions,* C.R.Acad.Sci.Paris, Ser.I 338, 759–762 (2004).

[27] J. Mahnkopf, *Cohomology of arithmetic groups, parabolic subgroups and the special values of L-functions of* GL_n, J. Inst. Math. Jussieu 4, no. 4, 553–637 (2005).

[28] C.J. Moreno and F. Shahidi, *The L-functions* $L(s, \mathrm{Sym}^m(r), \pi)$. Canad. Math. Bull. 28, no. 4, 405–410 (1985).
[29] V.K. Murty and D. Ramakrishnan, *Period relations and the Tate conjecture for Hilbert modular surfaces.* Invent. Math. 89, no. 2, 319–345 (1987).
[30] J. Neukirch, *Algebraic number theory.* Translated from the 1992 German original and with a note by Norbert Schappacher. With a foreword by G. Harder. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 322. Springer-Verlag, Berlin. xviii+571 pp (1999).
[31] T. Orloff, *Special values and mixed weight triple products (with an appendix by Don Blasius),* Invent. Math., 90, no. 1, 169–188, (1987).
[32] A.A. Panchishkin, *Motives over totally real fields and p-adic L-functions.* Ann. Inst. Fourier (Grenoble) 44, no. 4, 989–1023 (1994).
[33] A. Raghuram and F. Shahidi, *Special values of symmetric power L-functions,* In preparation.
[34] C.S. Rajan, *On the non-vanishing of the first Betti number of hyperbolic three manifolds.* Math. Ann. 330, no. 2, 323–329 (2004).
[35] D. Ramakrishnan, *Modularity of the Rankin-Selberg L-series, and multiplicity one for* $\mathrm{SL}(2)$. Ann. of Math. (2) 152, no. 1, 45–111 (2000).
[36] D. Ramakrishnan, *Algebraic cycles on Hilbert modular fourfolds and poles of L-functions.* Algebraic groups and arithmetic, 221–274, Tata Inst. Fund. Res., Mumbai, (2004).
[37] D. Ramakrishnan and S. Wang, *A cuspidality criterion for the functorial product on* $\mathrm{GL}(2) \times \mathrm{GL}(3)$ *with a cohomological application.* Int. Math. Res. Not., no. 27, 1355–1394 (2004).
[38] A.J. Scholl, *Motives for modular forms.* Invent. Math. 100, no. 2, 419–430 (1990).
[39] J. Schwermer, *Cohomology of arithmetic groups, automorphic forms and L-functions.* Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989), 1–29, Lecture Notes in Math., 1447, Springer, Berlin, 1990.
[40] G. Shimura, *On the holomorphy of certain Dirichlet series.* Proc. London Math. Soc. (3) 31, no. 1, 79–98 (1975).
[41] G. Shimura, *The special values of the zeta functions associated with cusp forms,* Comm. Pure Appl. Math., 29, no. 6, 783–804, (1976).
[42] G. Shimura, *On the periods of modular forms,* Math. Ann., 229, no. 3, 211–221, (1977).
[43] J. Sturm, *Special values of zeta functions, and Eisenstein series of half integral weight,* Amer. J. Math., 102, no. 2, 219–240, (1980).
[44] J. Sturm, *Evaluation of the symmetric square at the near center point,* Amer. J. Math., 111, no. 4, 585–598, (1989).
[45] J.-L. Waldspurger, *Quelques propriétés arithétiques de certaines formes automorphes sur* $\mathrm{GL}(2)$. Compositio Math. 54, no. 2, 121–171 (1985).
[46] D. Zagier, *Modular forms whose Fourier coefficients involve zeta functions of quadratic fields.* Modular forms of one variable VI, Springer Verlag LNM 627, 105–169 (1977).

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